Norwegian University of Science and Technology
Department of Engineering Cybernetics

Exercise 2
MILP algorithms and modeling techniques

Problem 1

Consider the linear integer program (IP)

$$J^* = \min_y \{c^T y : y \in X\}, \tag{1}$$

where X is the set of feasible integer points

$$X = \{y \in \mathbb{Z}_+^p : By \geq d\}. \tag{2}$$

and B is an $m \times p$ matrix. We denote as usual $P = \{y \in \mathbb{R}_+^p : By \geq d\}$ as the LP formulation for X obtained by relaxing the integrality restriction on y, with $X \subseteq P$. The *separation problem* as shown in lecture 2 is a general definition for the problem of generating a valid inequality $\pi y \leq \pi_0$ from a family of valid inequalities, in order to *cut off* a fractional LP solution from P. See Pochet and Wolsey (2006, p. 102) with extensions to MILPs. One of these families is the Chvátal-Gomory (CG) cuts, which were shown in lecture 2 to be inequalities of the form

$$\lfloor u^T B \rfloor y \leq \lfloor u^T d \rfloor, \tag{3}$$

where $u \in \mathbb{R}_m^+$ is a vector of multipliers, and $\lfloor \cdot \rfloor$ denotes the largest integer less than or equal to its argument. Consequently, the CG separation problem for (1) is defined as follows:

Definition 1. Given any point $\hat{y} \in P$, find (if any) a CG cut that is violated by $\hat{y}$, i.e. find a multiplier $u \in \mathbb{R}_m^+$ such that $\lfloor u^T B \rfloor \hat{y} > \lfloor u^T d \rfloor$, or prove that no such u exists.

a) The CG separation problem described above can be formulated as the MILP (Bonami et al., 2006)

$$\begin{aligned} \max_{\pi,u} \quad & \pi^T \hat{y} - \pi_0 & \\ \text{s.t.} \quad & \pi_i \leq u^T b_i, & \forall i = 1, \dots, p \\ & \pi_0 + 1 - \epsilon \geq u^T d, & \\ & u_j \geq 0, & \forall j = 1, \dots, m \\ & \pi_i \text{ integer}, & \forall i = 0, \dots, p \end{aligned} \tag{4}$$

where $\hat{y}$ is the (input) fractional LP solution we want to cut off, b_i for $i = 1, \dots, p$ are the columns of matrix B, and ϵ is a small nonnegative constant. Go through problem (4) in detail and explain why solving this MILP gives a CG cut.

b) At the end of definition 1, it says "*....or prove that no such u exists.*". How is this "property" of the separation problem retained when solving (4) to generate CG cuts?

c) In lecture 2, we considered the IP

$$\begin{aligned} \max_{y} \quad & 4y_1 - y_2 \\ \text{s.t.} \quad & -7y_1 - y_2 \leq 14 \\ & y_2 \leq 3 \\ & 2y_1 - 2y_2 \leq 3 \\ & y \in \mathbb{Z}_+^2 \end{aligned} \tag{5}$$

It was shown that using the multipliers $u^T = [0.571, 0.143, 0]$ from the LP relaxation and the rounding procedure to obtain a CG cut result in the valid inequality $4y_1 - y_2 \leq 8$, cutting off the LP solution $\hat{y} = (2.86, 3)$. Implement the CG separation problem (4) for the IP (5) in e.g. `Matlab` using YALMIP, with input $\hat{y} = (2.86, 3)$. Why do we obtain a different CG cut when solving(4)?

d) Draw the feasible region for (5), together with the cut (valid inequality) obtained when solving (5) and the cut $4y_1 - y_2 \leq 8$. Which of the cuts are strongest? What more can you say about the cut obtained when solving (4)?

Problem 2

Consider the 0-1 knapsack problem

$$\begin{aligned} J^* = \max_{y} \quad & 17y_1 + 10y_2 + 25y_3 + 17y_4 \\ \text{s.t.} \quad & 5y_1 + 3y_2 + 8y_3 + 7y_4 \leq 12 \\ & y \in \mathbb{B}^4 \end{aligned} \tag{6}$$

with optimal solution $y^* = (0, 1, 1, 0)$.

a) Solve (6) using branch-and-bound with

(i) Best-bound as node-selection strategy.

(ii) Depth-first as node-selection strategy.

Which of these two node-selection strategies give fewest number of nodes explore to prove optimality? Which one gives the first integer solution?

b) Consider the inequalities

$$y_3 + y_4 \leq 1 \tag{7a}$$
$$y_1 + y_3 \leq 1 \tag{7b}$$

Are these valid inequalities for (6) (hint: set y_3 and y_4 to one a check for feasibility)? What happens with the number of nodes needed in the BB tree if you add these inequalities a priori to (6)?

References

Bonami, P., Cornuéjols, G., Dash, S., Fischetti, M., and Lodi, A. (2006). Projected Chvátal-Gomory cuts for mixed integer linear programs. *Mathematical Programming*, 113(2):241–257.

Pochet, Y. and Wolsey, L. A. (2006). *Production Planning by Mixed Integer Programming.* Springer, New York.