Voluntary assignment in TTK16: Optimization in Energy and Oil & Gas Systems

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In this assignment, we will consider variations of the facility location problem shown in the first MILP lecture:

> Suppose we have m clients, indexed by i, that are to be served by facilities than can be opened at n potential cites (locations), indexed by j. Supplying client i's demand from a facility at location j gives a profit c_{ij}, while there is a cost d_j to open a facility at location j.
>
> Let $y_j = 1$ if facility j is opened, and $y_j = 0$ otherwise. Further, let x_{ij} be the *fraction* of client i's demand that is served by facility j. The problem consists of choosing optimal facility locations and assigning clients to these facilities.

We will consider the *uncapacitated* facility location problem (UFL), meaning that there is no limit on the number of clients a facility can serve. The objective function for the UFL is given by

$$\max_{x,y} \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} - \sum_{j=1}^{n} d_j y_j \tag{1}$$

and the satisfaction of demand for each client i given by the constraint

$$\sum_{j=1}^{n} x_{ij} = 1, \qquad i = 1 \dots m \tag{2}$$

$$y_j \in \{0, 1\} \tag{3}$$

We will start by considering different formulations for the constraints ensuring that only opened facilities can serve clients.

1. Explain why $\sum_{i=1}^{m} x_{ij} \leq m$ <u>if</u> a facility j is open, and how this can be used to formulate a reduced-size but *weak formulation* of the UFL:

$$\max_{x,y} \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} - \sum_{j=1}^{n} d_j y_j \tag{4a}$$

$$\text{s.t.} \tag{4b}$$

$$\begin{aligned}
&\sum_{j=1}^{n} x_{ij} = 1, && i = 1\dots m \\
&\sum_{i=1}^{m} x_{ij} \leq m y_j, && j = 1\dots n \\
&x_{ij} \geq 0, && i = 1\cdots n,\ j = 1\dots m
\end{aligned} \tag{4c}$$

$$y_j \in \{0,1\} \tag{4d}$$

2. For a given j, let the Boolean Y_j symbolize whether a facility is opened. To describe that a facility can only serve clients if it is opened, we can use the propositional logic

$$Y_j \Rightarrow 0 \leq x_{ij} \leq 1, \qquad i = 1,\dots,m. \tag{5}$$

The logical proposition (5) can be rewritten as the disjunction

$$\begin{bmatrix} Y_j \\ 0 \leq x_{1j} \leq 1 \\ 0 \leq x_{2j} \leq 1 \\ \vdots \\ 0 \leq x_{mj} \leq 1 \end{bmatrix} \veebar \begin{bmatrix} \neg Y_j \\ x_{1j} = 0 \\ x_{2j} = 0 \\ \vdots \\ x_{mj} = 0 \end{bmatrix} \tag{6}$$

The convex hull of linear disjunctions (see presentation from the first session),

$$[ax \leq b] \veebar [dx \leq e] \tag{7}$$

where $x \in \mathbb{R}$ with lower and upper bound, $0 \leq x \leq U$, are given by

$$x = z^1 + z^2 \tag{8a}$$
$$az^1 \leq by^1 \tag{8b}$$
$$dz^2 \leq ey^2 \tag{8c}$$
$$y^1 + y^2 = 1, \quad y^1, y^2 \in \{0,1\} \tag{8d}$$
$$0 \leq z^k \leq Uy^k, \quad k = 1,2 \tag{8e}$$

Show that by using (7)–(8), the convex hull reformulation of (6) leads to the tight setup-forcing constraints

$$x_{ij} \leq y_j, \qquad i = 1\dots m,\ j = 1\dots n \tag{9}$$

3. Show that the strong forcing constraints (9) implies the weak forcing constraints (4c), but not the opposite way. Consider a case with $n = 1$ facilities and $m = 2$ clients, and plot or draw the strong and weak forcing constraints. Why is (4) a weak formulation? What is expected to be gained by using the strong rather than the weak formulation?

4. Implement the UFL using either the provided GAMS code, or some other programming language. For GAMS:

   (i) Download a demo-version of GAMS from `www.gams.com`. Follow the installation instructions.

   (ii) Save and extract the provided files in a new folder.

   (iii) Create a project with name e.g. *TTK16Oving* in the same folder as you put the provided files.

   Try different setups of m and n with the strong and weak forcing constraints, respectively, and observe the difference in solution of the LP relaxation and the MILP using the to formulations. Explain. Try turning of cut-generation by using the provided option file in GAMS, and see how this impacts the number of nodes (displayed next to the MIP solution). You may use the provided `Matlab` script to generate profits c_{ij} and costs d_j for different setups of the problem.

5. Why does the strong formulation of the UFL provide solutions that are nearly integer, but not completely? What are the necessary steps to ensure that the LP relaxation provides integer feasible solutions? Would it make sense to perform these steps in practice rather than using a branch-and-cut code?

6. How would the constraints in the facility location problem be modified if each facility has a capacity u_j and each client a demand b_i?