Introduction to Integer Programming

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Summary

What is an Integer Problem?

An integer problem can be expressed as:

$$\begin{aligned} PL: \quad & \max \ c^{\mathrm{T}} x \\ & \text{s.t. } Ax \le b, \\ & \qquad x \ge 0 \end{aligned}$$

where:

- $x \in \mathbb{R}^{n \times 1}$
- $A \in \mathbb{R}^{m \times n}$
- $c \in \mathbb{R}^{n \times 1}$
- $b \in \mathbb{R}^{m \times 1}$

Integer Problems

There are several classes of integer problems

- Integer (Linear) Problem
- Mixed-Integer (Linear) Problem
- Linear Binary Problem
- Combinatorial Optimization Problem

Integer Problem

Integer Linear Problem (IP):

$$
\begin{aligned}
PL: \quad \max\ & c^{\mathrm{T}} x \\
\text{s.t. } & Ax \leq b, \\
& x \in \mathbb{Z}^n
\end{aligned}
$$

Mixed-Integer (Linear) Problem

Mixed-Integer Linear Problem (MILP):

$$
\begin{aligned}
PL: \quad \max\ & c^{\mathrm{T}} x \\
\text{s.t. } & Ax \leq b, \\
& x = (x_{\mathrm{C}}, x_{\mathrm{I}}) \\
& x_{\mathrm{C}} \geq 0 \\
& x_{\mathrm{I}} \in \mathbb{Z}^n
\end{aligned}
$$

Summary

Rounding and Integer Programming

Questão

- Why not use Linear Programming?
- We could disconsider the constraints on binary variables.
- Obtain an optimal solution $x^\star$ for the resulting linear program.
- An then round $x^\star$ such as to obtain a solution to the integer program.

Rounding and Integer Programming

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Rounding and Integer Programming

Issue

- The rounding strategy does not work.
- The following counter-example clarifies the issue:

$$\begin{array}{llcrcr} \max & x_1 & + & 0.6x_2 & & \\ \text{s.t.}: & 50x_1 & + & 31x_2 & \leq & 250 \\ & 3x_1 & - & 2x_2 & \geq & -4 \end{array}$$

with being $x_1, x_2 \geq 0$ and integer.

- An optimal solution to LP, $x_{PL} = (\frac{376}{193}, \frac{950}{193}) = (1.94, 4.92)$, could be rounded to obtain the solution $\overline{x}_{PL} = (2, 4)$.
- But this solution is quite "*far*" from the optimal solution $x^\star = (5, 0)$.

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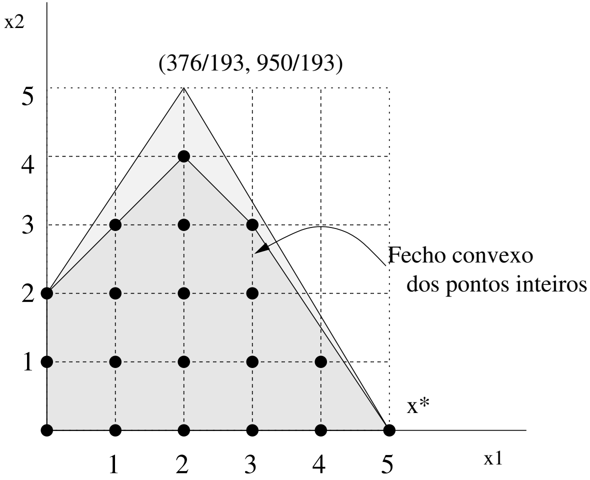
Rounding and Integer Programming



Summary

Fundamentals of Integer Programming

Applications

- Several poblems of academic and practical relevance can be formulade in integer programming.
- Examples:
  - combinatorial problems found in graph theory;
  - problems in logic; and
  - problems in logistics.

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Fundamentals of Integer Programming

Airline Crew Scheduling

- Allocation of flight crews subject to physical, temporal, and work-related constraints.
- High economic impact on airline companies.
- Given flight legs for a type of airplane, the problem is to allocate weekly crews to cyclic flight routes.

Travel Salesman Problem

Background

- Choose an order for a travel salesman to leave his home city, let us say city 1, visit the remaining $n-1$ cities precisely once, and then return to the home city.
- The distance traveled should be as short as possible.
- We are given set of n cities.
- c_{ij} is the cost o (distance) to travel from city i to city j.

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Travel Salesman Problem

Background

- The problem is to find the shortest route (circuit) that visits each city precisely once and whose travel distance is minimum.
- Application are found in vehicle routing, welding of electronic circuits, and garbage collection.

Travel Salesman Problem

Defining variables

$$x_{ij} = \begin{cases} 1 & \text{if salesman travels from city } i \text{ to city } j \\ 0 & \text{otherwise} \end{cases}$$

Travel Salesman Problem

Defining constraints

a) The salesman departs from city i exactly once:

$$\sum_{j=1}^{n} x_{ij} = 1 \qquad i = 1, \ldots, n$$

b) The salesman arrives at city j exactly once:

$$\sum_{i=1}^{n} x_{ij} = 1 \qquad j = 1, \ldots, n$$

Travel Salesman Problem

Defining constraints

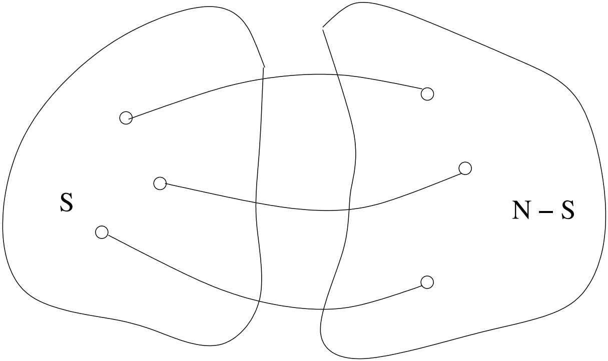
c) Connectivity constraints:

$$\sum_{i \in S} \sum_{j \notin S} x_{ij} \geq 1 \qquad \forall S \subset N,\ S \neq \emptyset$$

or subtour elimination:

$$\sum_{i \in S} \sum_{j \in S,\, j \neq i} x_{ij} \leq |S| - 1 \qquad \forall S \subseteq N,\ 2 \leq |S| \leq n - 1$$

Travel Salesman Problem



Travel Salesman Problem

Defining the objective

$$\min \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$

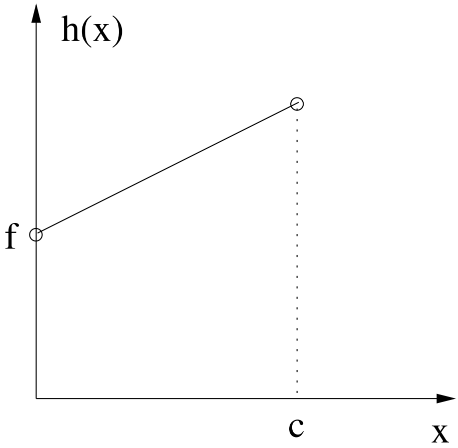
Summary

Modeling Fixed Cost

We wish to model the nonlinear function given by:

$$h(x) = \begin{cases} f + px & \text{if } 0 < x \leq c \\ 0 & \text{if } x = 0 \end{cases}$$

Modeling Fixed Cost



Modeling Fixed Cost

- Variáveis:

$$y = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$$

- Constraints and objective function:

$$h(x) = fy + px$$
$$x \leq cy$$
$$y \in \{0, 1\}$$

Model valid only for minimization.

Discrete Alternatives and Disjunctions

- A promising area in the theory and practice is disjunctive programming, that is, models and algorithms based on disjunctions.
- To understand disjunctive programming, suppose that $x \in \mathbb{R}^n$ satisfies:

$$\begin{gathered} 0 \leq x \leq u \text{ e} \\ a_1^T x \leq b_1 \text{ ou } a_2^T x \leq b_2 \end{gathered} \tag{1}$$

x must satisfy another linear constraints, not being necessary that it will satisfy both constraints.

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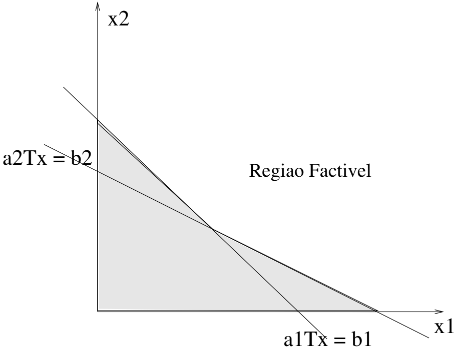
Discrete Alternative and Disjunctions

The feasible region of a disjunction with two constraints: notice that the feasible region is nonconvex.



Discrete Alternatives and Disjunctions

- How do we represent the disjunction (1) in mixed-integer linear programming.
- Let $M = \max_{i=1,2}\{a_i^T x - b_i : 0 \leq x \leq u\}$.
- Fist, we introduce two binary variables, y_1 and y_2, whose semantics is explained below:

$$y_1 = \begin{cases} 1 & \text{if } x \text{ satisfies } a_1^T x \leq b_1 \\ 0 & \text{otherwise} \end{cases}$$

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Discrete Alternatives and Disjunctions

Given the above variables, we can introduce the complete formulation:

$$
\begin{gathered}
a_1^{\mathrm{T}} x \leq b_1 + M(1 - y_1) \\
a_2^{\mathrm{T}} x \leq b_2 + M(1 - y_2) \\
y_1 + y_2 = 1 \\
y_1, y_2 \in \{0, 1\} \\
0 \leq x \leq u
\end{gathered}
$$

Discrete Alternatives and Disjunctions

- Disjunctions appear in scheduling problem.
- Tasks 1 and 2 must be processed in a given machine, but not simultaneously.
- Let p_i be the processing time of task i and t_i the time processing begins.
- Then, we can express temporal precedence of one task in relation to the other by a disjunction:

$$t_1 + p_1 \leq t_2 \text{ or } t_2 + p_2 \leq t_1$$

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Power of Binary Variables

- The power function x^p, $p \in \mathbb{N}_+$, with $x \in \{0, 1\}$ is nonlinear.
- Notice that $x^p = x$ since:
  - $x^p = 0$ if $x = 0$ and
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Product of Binary Variables

- Consider the term $y = x_1 x_2 x_3$, in which $x_i \in \{0, 1\}$.
- The nonlinear term can be reformulated as:

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\begin{aligned}
y &\le x_1 \\
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y &\ge x_1 + x_2 + x_3 - 2 \\
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Sign Function: $\text{sign}(x)$

- The function $\text{sign}(\cdot)$ can be modeled using a binary variable.
- Assuming that $|x| \leq M$, then:

$$\begin{aligned} x &\leq Mz, \\ x &\geq -M(1-z), \\ \text{sign}(x) &= (2z-1), \\ z &\in \{0,1\} \end{aligned}$$

Some Challenges

Can you model the following functions in mixed-integer linear programming?

- $y = \max\{x_1, x_2\}$?
- $y = |x|$?

Integer Programming

- Thank you for attending this lecture!!!