

Introduction to Integer Programming

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Introduction

Rounding and Integer Programming

Applications

Modeling Strategies

Summary

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Integer Problem

Integer Linear Problem (IP):

$$\begin{aligned} PL : \quad & \max \quad c^T x \\ & \text{s.t.} \quad Ax \leq b, \\ & \quad \quad x \in \mathbb{Z}^n \end{aligned}$$

Mixed-Integer (Linear) Problem

Mixed-Integer Linear Problem (MILP):

$$\begin{aligned} IP : \quad & \max \quad c^T x \\ & \text{s.t.} \quad Ax \leq b, \\ & \quad \quad x = (x_C, x_I) \\ & \quad \quad x_C \geq 0 \\ & \quad \quad x_I \in \mathbb{Z}^n \end{aligned}$$

Integer Problems

There are several classes of integer problems

- ▶ Integer (Linear) Problem
- ▶ Mixed-Integer (Linear) Problem
- ▶ Linear Binary Problem
- ▶ Combinatorial Optimization Problem

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Rounding and Integer Programming

Questão: Why not use Linear Programming?

- ▶ We could disregard the constraints on binary variables.
- ▶ Obtain an optimal solution x^* for the resulting linear program.
- ▶ And then round x^* such as to obtain a solution to the integer program.

Rounding and Integer Programming

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Issue

- ▶ The rounding strategy does not work.
- ▶ The following counter-example clarifies the issue:

$$\begin{array}{llllll} \max & x_1 & + & 0.6x_2 & & \\ \text{s.t. :} & 50x_1 & + & 31x_2 & \leq & 250 \\ & 3x_1 & - & 2x_2 & \geq & -4 \end{array}$$

with being $x_1, x_2 \geq 0$ and integer.

Rounding and Integer Programming

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with being $x_1, x_2 \geq 0$ and integer.

- ▶ An optimal solution to LP, $x = (\frac{376}{193}, \frac{950}{193}) = (1.94, 4.92)$, could be rounded to obtain the solution $\bar{x} = (2, 4)$.
- ▶ But this solution is quite “*far*” from the optimal solution $x^* = (5, 0)$.

Rounding and Integer Programming

Issue

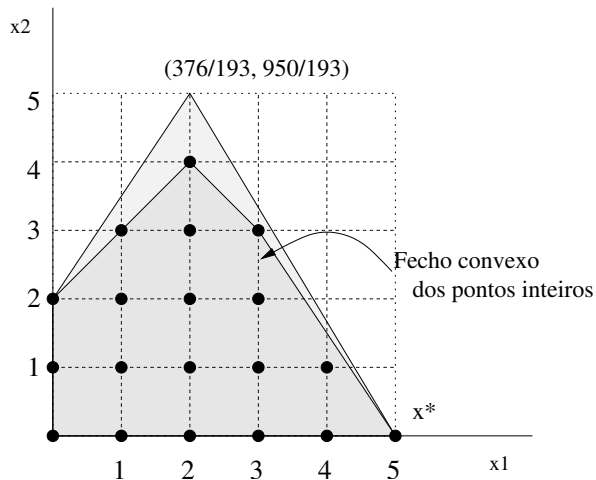
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Rounding and Integer Programming



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Fundamentals of Integer Programming

Applications

- ▶ Several problems of academic and practical relevance can be formulated in integer programming.
- ▶ Examples:
 - ▶ combinatorial problems found in graph theory (set covering, maximum clique);
 - ▶ problems in logic (satisfiability problem); and
 - ▶ problems in logistics.

Fundamentals of Integer Programming

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 - ▶ problems in logistics.

Fundamentals of Integer Programming

Airline Crew Scheduling

- ▶ Allocation of flight crews subject to physical, temporal, and work-related constraints.
- ▶ High economic impact on airline companies.
- ▶ Given flight legs for a type of airplane, the problem is to allocate weekly crews to cyclic flight routes.

Travel Salesman Problem

Background

- ▶ Choose an order for a travel salesman to leave his home city, let us say city 1, visit the remaining $n - 1$ cities precisely once, and then return to the home city.
- ▶ The distance traveled should be as short as possible.
- ▶ We are given a set of n cities.
- ▶ c_{ij} is the cost (distance) to travel from city i to city j .

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Travel Salesman Problem

Background

- ▶ The problem is to find the shortest route (circuit) that visits each city precisely once and whose travel distance is minimum.
- ▶ Applications are found in vehicle routing, welding of electronic circuits, and garbage collection.

Travel Salesman Problem

Defining variables

$$x_{ij} = \begin{cases} 1 & \text{if salesman travels from city } i \text{ to city } j \\ 0 & \text{otherwise} \end{cases}$$

Travel Salesman Problem

Defining constraints

- a) The salesman departs from city i exactly once:

$$\sum_{j=1}^n x_{ij} = 1 \quad i = 1, \dots, n$$

- b) The salesman arrives at city j exactly once:

$$\sum_{i=1}^n x_{ij} = 1 \quad j = 1, \dots, n$$

Travel Salesman Problem

Defining constraints

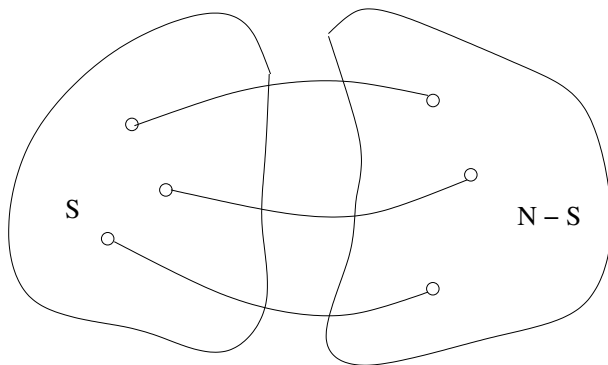
c) Connectivity constraints:

$$\sum_{i \in S} \sum_{j \notin S} x_{ij} \geq 1 \quad \forall S \subset N, S \neq \emptyset$$

or subtour elimination:

$$\sum_{i \in S} \sum_{j \in S, j \neq i} x_{ij} \leq |S| - 1 \quad \forall S \subseteq N, 2 \leq |S| \leq n - 1$$

Travel Salesman Problem



Travel Salesman Problem

Defining the objective

$$\min \sum_{i=1}^n \sum_{j=1}^n c_{ij} x_{ij}$$

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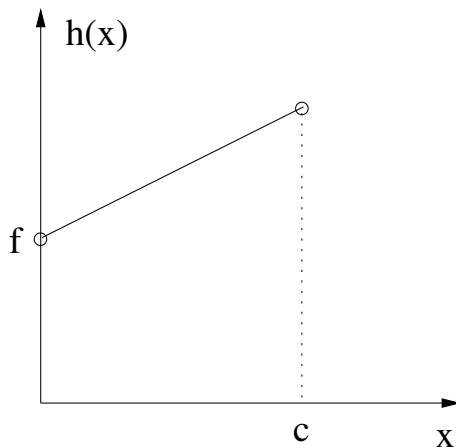
Modeling Strategies

Modeling Fixed Cost

We wish to model the nonlinear function given by:

$$h(x) = \begin{cases} f + px & \text{if } 0 < x \leq c \\ 0 & \text{if } x = 0 \end{cases}$$

Modeling Fixed Cost



Modeling Fixed Cost

- Variables:

$$y = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$$

- Constraints and objective function:

$$\begin{aligned} h(x) &= fy + px \\ x &\leq cy \\ y &\in \{0, 1\} \end{aligned}$$

- Model valid only for minimization.

Discrete Alternatives and Disjunctions

- ▶ A promising area in theory and practice is disjunctive programming, that is, models and algorithms based on disjunctions.
- ▶ To understand disjunctive programming, suppose that $x \in \mathbb{R}^n$ satisfies:

$$0 \leq x \leq u \text{ and } (a_1^T x \leq b_1) \text{ or } (a_2^T x \leq b_2) \quad (1)$$

- ▶ x must satisfy one of the linear constraints, not necessarily both constraints.

Discrete Alternatives and Disjunctions

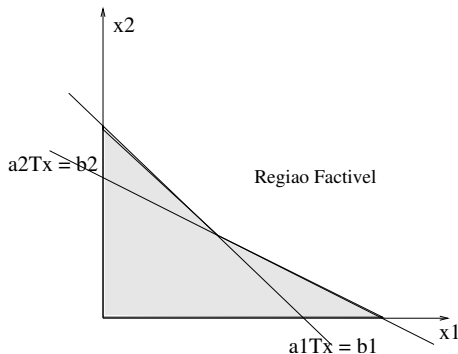
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Discrete Alternatives and Disjunctions

The feasible region of a disjunction with two constraints: notice that the feasible region is nonconvex.



Discrete Alternatives and Disjunctions

How do we represent the disjunction (1) in mixed-integer linear programming.

- ▶ Let $M = \max_{i=1,2} \{a_i^T x - b_i : 0 \leq x \leq u\}$.
- ▶ First, we introduce two binary variables, y_1 and y_2 , whose semantics is explained below:

$$y_1 = \begin{cases} 1 & \text{if } x \text{ satisfies } a_1^T x \leq b_1 \\ 0 & \text{otherwise} \end{cases}$$

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Discrete Alternatives and Disjunctions

Given the above variables, we can introduce the complete formulation:

$$a_1^T x \leq b_1 + M(1 - y_1)$$

$$a_2^T x \leq b_2 + M(1 - y_2)$$

$$y_1 + y_2 = 1$$

$$y_1, y_2 \in \{0, 1\}$$

$$0 \leq x \leq u$$

Discrete Alternatives and Disjunctions

- ▶ Disjunctions appear in scheduling problems.
- ▶ Tasks 1 and 2 must be processed in a given machine, but not simultaneously.
- ▶ Let p_i be the processing time of task i and t_i the time processing begins.
- ▶ Then, we can express temporal precedence of one task in relation to the other by a disjunction:

$$(t_1 + p_1 \leq t_2) \text{ or } (t_2 + p_2 \leq t_1)$$

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Power of Binary Variables

- ▶ The power function x^p , $p \in \mathbb{N}_+$, with $x \in \{0, 1\}$ is nonlinear.
- ▶ Notice that $x^p = x$ since:
 - ▶ $x^p = 0$ if $x = 0$ and
 - ▶ $x^p = 1$ and $x = 1$.
- ▶ Thus, it is possible to linearize the term x^p .

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Product of Binary Variables

- ▶ Consider the term $y = x_1x_2x_3$, in which $x_i \in \{0,1\}$.
- ▶ The nonlinear term can be reformulated as:

$$y \leq x_1$$

$$y \leq x_2$$

$$y \leq x_3$$

$$y \geq x_1 + x_2 + x_3 - 2$$

$$y \geq 0$$

$$x_1, x_2, x_3 \in \{0,1\}$$

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Sign Function: $\text{sign}(x)$

- ▶ The function $\text{sign}(\cdot)$ can be modeled using a binary variable.
- ▶ Assuming that $|x| \leq M$, then:

$$x \leq Mz,$$

$$x \geq -M(1 - z),$$

$$\text{sign}(x) = (2z - 1),$$

$$z \in \{0, 1\}$$

Some Challenges

Can you model the following functions in mixed-integer linear programming?

- ▶ $y = \max\{x_1, x_2\}$?
- ▶ $y = |x|$?

Integer Programming

- ▶ Thank you for attending this lecture!!!