Linear Programming

Fundamentals, Algorithm Simplex and Duality

Eduardo Camponogara

Department of Automation and Systems Engineering
Federal University of Santa Catarina

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Sumário

History of Linear Programmming

History

- Linear Programming was incepted in the 1930's, experience great advance in the 1940's with the development of the Smiplex Algorithm by George Dantzig.
- LP algorithms are rather efficient.
- Specially-tailored algorithms were proposed, particularly for network-flow problems.

History of Linear Programming

History

- In 1979, Khachiyan discovered the first polynomial-time algorithms, which became known as the ellipsoid algorithm.
  - It is not practical.
  - The ellipsoid algorithm has significant impacts on integer programming and in the solution of linear programming.

History of Linear Programming

History (continued)

- In 1984, Karmakar discovered the an interior-point algorithm with polynomial time.
- This discovered led to breakthroughs in quadratic programming and convex optimization, among other fields.
- Despite the polynomial-time of the interio-point method, the Simplex algorithm remains widely used.

THe Linear Programming Problem

Programa Linear:

$$\begin{aligned} \max\ & c^{\mathrm{T}} x \\ \text{s.t. } & Ax \leq b \\ & x \geq 0 \end{aligned}$$

Transforming $\leq$ into $=$

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n \geq b$$
$$\Leftrightarrow$$
$$\begin{cases} a_1x_1 + a_2x_2 + \ldots + a_nx_n - s = b \\ s \geq 0 \end{cases}$$

Here, s is a (*slack variable*) since its value corresponds to the amount resource b which is not used.

Dealing with Real Variables

We can convert sign-unconstrained variables into nonnegative variables:

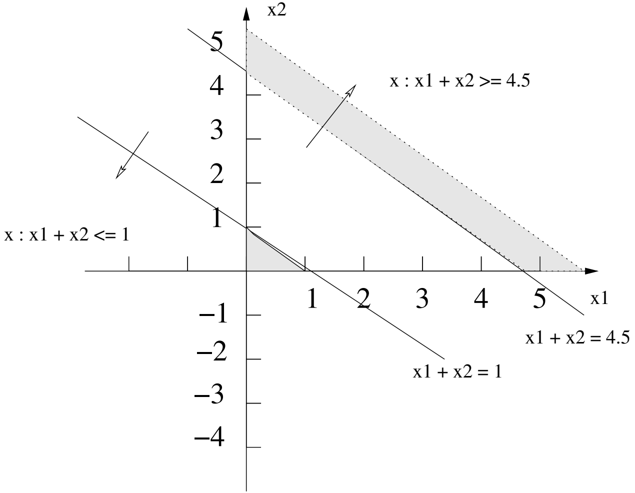
$$x \in \mathbb{R} \Leftrightarrow \begin{cases} x = x^+ - x^- \\ x^+ \geq 0 \\ x^- \geq 0 \end{cases}$$

Infeasible Problem

- A problem is said to be infeasible if there does not exist a candidate solution $x \in \mathbb{R}^n$ that meets all constraints.
- The problem below is infeasible since $S = \{x \in \mathbb{R}^n : Ax \leq b, x \geq 0\} = \emptyset$.
- Example of infeasible problem:

$$
\begin{array}{ll}
\max & 5x_1 + 4x_2 \\
s.t.: & \\
& \begin{array}{rcrcr} x_1 & + & x_2 & \leq & 1 \\ -2x_1 & - & 2x_2 & \leq & -9 \end{array} \\
& x_1, x_2 \geq 0
\end{array}
$$

Infeasible Problem



Unbounded Problem

- A problem is unbounded if there does not exist an upper bound on the value of the objective.
- In other words, the objective value can grow arbitrarily large.
- An example of unbounded problem:

$$
\begin{array}{ll}
\max & x_1 - 4x_2 \\
\text{s.t. :} & \\
& \begin{array}{rcrcr} -2x_1 & + & x_2 & \leq & -1 \\ -x_1 & - & 2x_2 & \leq & -2 \end{array} \\
& x_1, x_2 \geq 0
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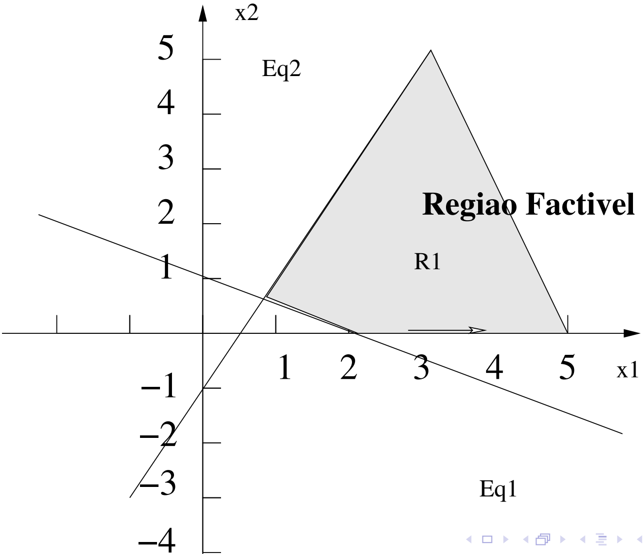
$$

Unbounded Problem



Sumário

Simplex Algorithm

History

- Proposed by George Dantzig in the 1940's.
- Today's algorithm differs from the original version.
- It is the building-block of branch-and-bound and branch-and-cut algorithms for integer programming.

Standard Problem

Linear Program:

$$\begin{aligned} \max\ & c^{\mathrm{T}} x \\ \text{s.t.}\ & Ax = b \\ & x \geq 0 \end{aligned}$$

Simplex Algorith,

History

- Simplex can be viewed as camobinatorial search for the optimal columns of constraints matrix: columns that induce an optimal basis.
- There is an exponential number of column combinations.
- Despite this large number, Simplex is efficient in practice.

Sample Problem

Consider the following linear program:

$$\begin{aligned} \max \quad & 5x_1 + 4x_2 + 3x_3 \\ \text{s.t.}: \quad & \\ & 2x_1 + 3x_2 + x_3 && \leq 5 \\ & 4x_1 + x_2 + 2x_3 && \leq 11 \\ & 3x_1 + 4x_2 + 2x_3 && \leq 8 \\ & x_1, x_2, x_3 \geq 0 \end{aligned} \tag{1}$$

Sample Problem

After introducing slack variables, the problema assumes the form:

$$\begin{array}{llcrcrcrcr} \max & \delta & = & 0 & + & 5x_1 & + & 4x_2 & + & 3x_3 \\ s.t.: & & & & & & & & & \\ & w_1 & = & 5 & - & 2x_1 & - & 3x_2 & - & x_3 \\ & w_2 & = & 11 & - & 4x_1 & - & x_2 & - & 2x_3 \\ & w_3 & = & 8 & - & 3x_1 & - & 4x_2 & - & 2x_3 \\ & & & \multicolumn{7}{c}{x_1, x_2, x_3, w_1, w_2, w_3 \geq 0} \end{array} \tag{2}$$

Sample Problem

Dictionary

- The system (2) cas the LP problem in a form known as "dictionary."
- In the "dictionary", the solution is given in terms of a subset o variables (*basic variables*), with cardinality equal to the number of constraints, which are given as a function of the remaining variables (*nonbasic variables*).
- The nonbasic variables assume value "zero."
- Thus, a solution can be obtained directly from the dictionary.

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Sample Problem

Dictionary (continued)

- In the above dictionary, the basis if formed by variables w_1, w_2 and w_3.
- The nonbasic variables are x_1, x_2 e x_3.
- Since nonbasic variables assume value zero, we obtain a solution from the basic variables: $w_1 = 5$, $w_2 = 11$ and $w_3 = 8$.
- The resulting solution is feasible:
  - All variables (basic and nonbasic) are nonnegative.
  - The objective function has value $\delta = 0$.

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Principles of Simplex

Iterative method

- Simplex is an iterative process that starts with a solution $y^0 = (x_1^0 \ \dots \ x_n^0 \ w_1^0 \ \dots \ w_m^0)^T$, in which $n = 3$ and $m = 3$, satisfying the equations of (1).
- Starting from (2), the Simplex seeks a new solution y^1 such that: $5x_1^1 + 4x_2^1 + 3x_3^1 > 5x_1^0 + 4x_2^0 + 3x_3^0$.
- To that end, it is necessary to bring a nonbasic variables, with a positive coefficient in the equation δ, into the basis.
- This will raise the value of the nonbasic variable entering the basis, thereby increasing the objective.

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Principles of Simplex

Iterative method (continued)

- Obviously, the variable that enters the basis cannot increase indefinitely, unless the problem is unbounded.
- The first basic variable to become zero must leave the basis, becoming nonbasic at the next iteration.
- The process is repeated until convergence to the optimum, or the problem is detected to be unbounded.

Principles of Simplex

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Simplex Algorithm: Example

Initialization

To start the iterative process, a feasible starting solution solution, such as:

$$x_1^0 = 0,\ x_2^0 = 0,\ x_3^0 = 0,\ w_1^0 = 5,\ w_2^0 = 11,\ w_3^0 = 8$$

This solution y^0 induces an objective value $\delta = 0$.

Simplex Algorithm: Example

Step 1

- The incumbent solution is not optimal.
- Any increase in the value of x_1 increases the o valor de δ.
- The value of x_1 cannot increase arbitrarily since it is limited by the inequalities:

$$\left\{\begin{array}{rcrcl} w_1 & = & 5 - 2x_1 & \geq & 0 \\ w_2 & = & 11 - 4x_1 & \geq & 0 \\ w_3 & = & 8 - 3x_1 & \geq & 0 \end{array}\right. \Rightarrow \left\{\begin{array}{rcrcr} x_1 & \leq & 5/2 & = & 2.5 \\ x_1 & \leq & 11/4 & = & 2.75 \\ x_1 & \leq & 8/3 & = & 2.667 \end{array}\right.$$

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Simplex Algorithm: Example

Step 1 (continued)

Thus, the value of x_1 at the next iteration should be less than $\{5/2, 11/4, 8/3\}$, which leads to:

$$x_1 = \frac{5}{2} - \frac{1}{2}w_1 - \frac{3}{2}x_2 - \frac{1}{2}x_3 \tag{3}$$

Simplex Algorithm: Example

Step 1 (continued)

Replacing (3) in the system (2) in order to move w_1 to the right-hand size, we obtain:

$$
\begin{aligned}
x_1 &= \tfrac{5}{2} - \tfrac{1}{2}w_1 - \tfrac{3}{2}x_2 - \tfrac{1}{2}x_3 \\[1em]
w_2 &= 11 - 4\left(\tfrac{5}{2} - \tfrac{1}{2}w_1 - \tfrac{3}{2}x_2 - \tfrac{1}{2}x_3\right) - x_2 - 2x_3 \\
&= 1 + 2w_1 + 5x_2 \\[1em]
w_3 &= 8 - 3\left(\tfrac{5}{2} - \tfrac{1}{2}w_1 - \tfrac{3}{2}x_2 - \tfrac{1}{2}x_3\right) - 4x_2 - 2x_3 \\
&= \tfrac{1}{2} + \tfrac{3}{2}w_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 \\[1em]
\delta &= 5\left(\tfrac{5}{2} - \tfrac{1}{2}w_1 - \tfrac{3}{2}x_2 - \tfrac{1}{2}x_3\right) + 4x_2 + 3x_3 \\
&= \tfrac{25}{2} - \tfrac{5}{2}w_1 - \tfrac{7}{2}x_2 + \tfrac{1}{2}x_3
\end{aligned}
\tag{4}
$$

Simplex Algorithm: Example

Step 1 (continued)

Now, replacing equations (4) in the "dictionary" (2), the following dictionary results:

$$\begin{array}{lrcrrrr} \max & \delta & = & \frac{25}{2} & -\frac{5}{2}w_1 & -\frac{7}{2}x_2 & +\frac{1}{2}x_3 \\ & x_1 & = & \frac{5}{2} & -\frac{1}{2}w_1 & -\frac{3}{2}x_2 & -\frac{1}{2}x_3 \\ & w_2 & = & 1 & +2w_1 & +5x_2 & \\ & w_3 & = & \frac{1}{2} & +\frac{3}{2}w_1 & +\frac{1}{2}x_2 & -\frac{1}{2}x_3 \end{array} \tag{5}$$

Simplex Algorithm: Example

Step 1 (continued)

- The solution induced by the dictionary (5) is $y^1 = (x_1^1, x_2^1, x_3^1, w_1^1, w_2^1, w_3^1) = (\frac{5}{2}, 0, 0, 0, 1, \frac{1}{2})$ with objective é $\delta = \frac{25}{2}$.
- In this dictionary, the variables x_1, w_2, e w_3 are basic such that the set $\mathbb{B} = \{x_1, w_2, w_3\}$ contains all basic variables.
- The remaining variables $\mathbb{N} = \{x_2, x_3, w_1\}$ form the set of nonbasic variables.

Simplex Algorithm: Example

Step 1 (continued)

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Simplex Algorithm: Example

Step 2

- The incumbent solution is not optimal!
- Notice that a small increase in the value of x_3 will invariable increase the value of δ.
- We cannot increase the value of x_3 indefinitely becasue it will turn the solution infeasible — other variables may become negative.

Simplex Algorithm: Example

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Simplex Algorithm: Example

Step 2 (continued)

For the solution to remain feasible, the following inequalities must be respected:

$$\begin{cases} x_1 & = & \frac{5}{2} - \frac{1}{2}x_3 & \geq & 0 \\ w_3 & = & \frac{1}{2} - \frac{1}{2}x_3 & \geq & 0 \end{cases} \Rightarrow \begin{cases} x_3 \leq 5 \\ x_3 \leq 1 \end{cases}$$

Simplex Algorithm: Example

Step 2 (continued)

- Therefore, w_3 must leave the basis so that x_3 enter the basis without violating any constraint.
- After replacing equation $x_3 = 1 + 3w_1 + x_2 - 2w_3$ in the equations of the dictionary (5), we obtain:

$$
\begin{aligned}
x_1 &= \tfrac{5}{2} - \tfrac{1}{2}w_1 - \tfrac{3}{2}x_2 - \tfrac{1}{2}(1 + 3w_1 + x_2 - 2w_3) \\
&= 2 - 2w_1 - 2x_2 + w_3 \\
\delta &= \tfrac{25}{2} - \tfrac{5}{2}w_1 - \tfrac{7}{2}x_2 + \tfrac{1}{2}(1 + 3w_1 + x_2 - 2w_3) \\
&= 13 - w_1 - 3x_2 - w_3
\end{aligned}
\qquad (6)
$$

Simplex Algorithm: Example

Step 2 (continued)

Replacing the equations of the dictionary (5) by the equations (6) results into a new dictionary:

$$
\begin{array}{lrcrrrr}
\text{Max} & \delta & = & 13 & -w_1 & -3x_2 & -w_3 \\
& x_1 & = & 2 & -2w_1 & -2x_2 & +w_3 \\
& w_2 & = & 1 & +2w_1 & +5x_2 & \\
& x_3 & = & 1 & +3w_1 & +x_2 & -2w_3
\end{array}
\qquad (7)
$$

whose basis is $\mathbb{B} = \{x_1, x_3, w_2\}$.

Simplex Algorithm: Example

Step 2 (continued)

The solution give by the dictionary (7) is optimal:

- $x_1 = 2$, $x_2 = 0$, $x_3 = 1$, $w_1 = 0$, $w_2 = 1$, $w_3 = 0$
- $\delta = 13$ is the optimal value because the coefficients of the nonbasic variables in the equation δ are all negative, in the dictionary given by (7),
- Increasing the value of any nonbasic variable will reduce the value of the objective.

Simplex Algorithm: Example

Step 2 (continued)

The solution give by the dictionary (7) is optimal:

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Sumário

Auxiliary Problem

Original Problem

$$
\begin{array}{ll}
\text{Maximize} & -2x_1 - x_2 \\
\text{Subject to}: & \\
& \begin{array}{rcrcr}
-x_1 & + & x_2 & \leq & -1 \\
-x_1 & - & 2x_2 & \leq & -2 \\
 & & x_2 & \leq & 1
\end{array} \\
& x_1, x_2 \geq 0
\end{array}
$$

Auxiliary Problem

Auxiliary Problem

$$
\begin{array}{lrcrcrcr}
\max & -x_0 & & & & & & \\
s.t.: & & & & & & & \\
 & -x_1 & + & x_2 & - & x_0 & \leq & -1 \\
 & -x_1 & - & 2x_2 & - & x_0 & \leq & -2 \\
 & & & x_2 & - & x_0 & \leq & 1 \\
 & x_0, x_1, x_2 \geq 0 & & & & & &
\end{array}
$$

Auxiliary Problem

Initial Dictionary

$$
\begin{array}{rrrrrrr}
\max & \delta & = & & & & -x_0 \\
& w_1 & = & -1 & +x_1 & -x_2 & +x_0 \\
& w_2 & = & -2 & +x_1 & +2x_2 & +x_0 \\
& w_3 & = & 1 & & -x_2 & +x_0
\end{array}
$$

Sumário

Duality

- Associated to any linear programming problem P (primal) is another linear programming probem, the dual, which is denoted D.
- Theoretical consequences.
- Any solution for the dual D induces a limit for the optimum value of the primal P, and vice versa.

Duality

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- Theoretical consequences.
- Any solution for the dual D induces a limit for the optimum value of the primal P, and vice versa.

Motivation

Consider the sample problem:

$$
\begin{array}{lll}
P: & \max & 4x_1 + x_2 + 3x_3 \\
 & \text{s.t.}: & \\
 & & x_1 + 4x_2 \leq 1 \\
 & & 3x_1 - x_2 + x_3 \leq 3 \\
 & & x_1, x_2, x_3 \geq 0
\end{array}
\tag{8}
$$

Any feasible solution to P induces a lower bound.

- For example, $x' = (1, 0, 0)$ shows that the optimum objective $\delta^\star \geq 4$.
- Using the solution $x'' = (0, 0, 3)$, we discover that $\delta^\star \geq 9$.

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Motivation

Issues

- Are these lower bounds close to the optimum?
- Let's multiply the first constraint of (8) by 2, multiply the second constraint by 3, and then add them up as follows:

$$\begin{array}{rcl} 2(x_1 + 4x_2) & \leq & 2(1) \\ 3(3x_1 - x_2 + x_3) & \leq & 3(3) \\ \hline 11x_1 + 5x_2 + 3x_3 & \leq & 11 \end{array}$$

- Notice that $4x_1 + x_2 + 3x_3 \leq 11x_1 + 5x_2 + 3x_3 \leq 11$, pois $x_j \geq 0$.
- Then we conclude that $9 \leq \delta^\star \leq 11$.

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Motivation

- To obtain tight bound, we follow the same procedure but, this time, we use vairables rather than fixed values.
- By multiplying the constraints with nonnegative variables:

$$\begin{array}{rcl} y_1(x_1 + 4x_2) & \leq & y_1 \\ y_2(3x_1 - x_2 + x_3) & \leq & 3y_2 \\ \hline (y_1 + 3y_2)x_1 + (4y_1 - y_2)x_2 + y_2x_3 & \leq & y_1 + 3y_2 \end{array}$$

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y_2(3x_1 - x_2 + x_3) & \leq & 3y_2 \\
\hline
(y_1 + 3y_2)x_1 + (4y_1 - y_2)x_2 + y_2x_3 & \leq & y_1 + 3y_2
\end{array}
$$

Motivation

Now, we stipulate that:

$$\begin{array}{lcl} y_1 + 3y_2 & \geq & 4 \\ 4y_1 - y_2 & \geq & 1 \\ y_2 & \geq & 3 \\ y_1, y_2 & \geq & 0 \end{array} \quad (9)$$

Values for (y_1, y_2) satisfying the inequalities (9) lead to the following inequalities:

$$\begin{array}{lcl} \delta & = & 4x_1 + x_2 + 3x_3 \\ & \leq & (y_1 + 3y_2)x_1 + (4y_1 - y_2)x_2 + y_2x_3 \\ & \leq & y_1 + 3y_2 \end{array}$$

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 & \leq & y_1 + 3y_2
\end{array}
$$

Motivation

- We obtain an upper bound $(y_1 + 3y_2)$ for $\delta^\star$.
- We wish to minimize this upper bound:

$$
\begin{array}{llll}
D: & \min & y_1 + 3y_2 & \\
 & \text{s.t.}: & & \\
 & & y_1 + 3y_2 & \geq 4 \\
 & & 4y_1 - y_2 & \geq 1 \\
 & & y_2 & \geq 3 \\
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\end{array}
$$

Motivation

- We obtain an upper bound $(y_1 + 3y_2)$ for $\delta^\star$.
- We wish to minimize this upper bound:

$$
\begin{array}{llll}
D: & \min & y_1 + 3y_2 & \\
 & \text{s.t.}: & & \\
 & & y_1 + 3y_2 & \geq 4 \\
 & & 4y_1 - y_2 & \geq 1 \\
 & & y_2 & \geq 3 \\
 & & y_1, y_2 & \geq 0
\end{array}
$$

Linear Programming

- Thank you for attending this lecture!!!