Linear Programming

Fundamentals, Algorithm Simplex and Duality

Eduardo Camponogara

Department of Automation and Systems Engineering
Federal University of Santa Catarina

October 2016

Introduction

Simplex Algorithm

Initial Solution

Duality

Summary

Introduction

Simplex Algorithm

Initial Solution

Duality

History of Linear Programming

History

- Linear Programming was incepted in the 1930's, experienced great advance in the 1940's with the development of the Simplex Algorithm by George Dantzig.
- LP algorithms are rather efficient.
- Specially-tailored algorithms were proposed, particularly for network-flow problems.

History of Linear Programming

History

- In 1979, Khachiyan discovered the first polynomial-time algorithm, which became known as the ellipsoid algorithm.
  - It is not practical.
  - The ellipsoid algorithm has significant impact on integer programming and in the solution of linear programming problems.

History of Linear Programming

History (continued)

- In 1984, Karmakar discovered an interior-point algorithm with polynomial time.
- This discovery led to breakthroughs in quadratic programming and convex optimization, among other fields.
- Despite the polynomial-time of the interior-point method, the Simplex algorithm remains widely used.

The Linear Programming Problem

Linear Program:

$$
\begin{aligned}
\max\ & c^{\mathrm{T}}x \\
\text{s.t. } & Ax \leq b \\
& x \geq 0
\end{aligned}
$$

Transforming $\geq$ into $=$

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n \geq b$$
$$\Leftrightarrow$$
$$\begin{cases} a_1x_1 + a_2x_2 + \ldots + a_nx_n - s = b \\ s \geq 0 \end{cases}$$

Here, s is a (*slack variable*) since its value corresponds to the amount of resource b which is not being used.

Dealing with Real Variables

We can convert sign-unconstrained variables into nonnegative variables:

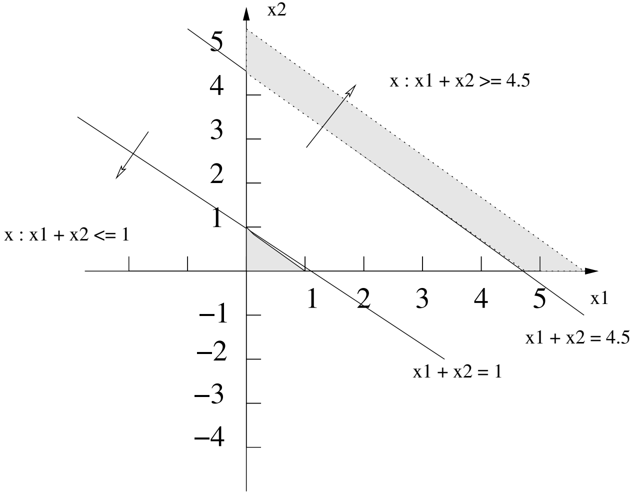
$$x \in \mathbb{R} \Leftrightarrow \begin{cases} x = x^+ - x^- \\ x^+ \geq 0 \\ x^- \geq 0 \end{cases}$$

Infeasible Problem

- A problem is said to be infeasible if there does not exist a candidate solution $x \in \mathbb{R}^n$ that meets all constraints.
- The problem below is infeasible since $S = \{x \in \mathbb{R}^n : Ax \leq b, x \geq 0\} = \emptyset$.
- Example of infeasible problem:

$$
\begin{array}{ll}
\max & 5x_1 + 4x_2 \\
s.t.: & \\
& \begin{array}{rcrcr}
x_1 & + & x_2 & \leq & 1 \\
-2x_1 & - & 2x_2 & \leq & -9
\end{array} \\
& x_1, x_2 \geq 0
\end{array}
$$

Infeasible Problem

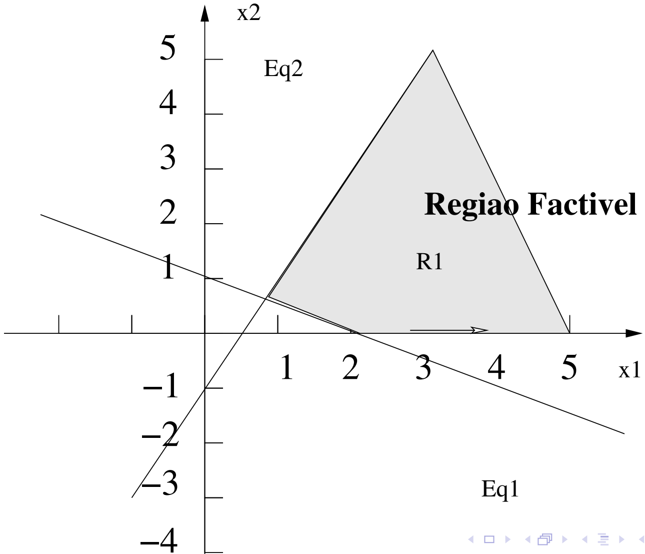


Unbounded Problem

- A problem is unbounded if there does not exist an upper bound on the value of the objective.
- In other words, the objective value can grow arbitrarily large.
- An example of unbounded problem:

$$\begin{array}{ll} \max & x_1 - 4x_2 \\ \text{s.t.}: & \\ & \begin{array}{rcrcr} -2x_1 & + & x_2 & \leq & -1 \\ -x_1 & - & 2x_2 & \leq & -2 \end{array} \\ & x_1, x_2 \geq 0 \end{array}$$

Unbounded Problem



Summary

Introduction

Simplex Algorithm

Initial Solution

Duality

Simplex Algorithm

History

- Proposed by George Dantzig in the 1940's.
- Today's algorithm differs from the original version.
- It is the building-block of branch-and-bound and branch-and-cut algorithms for integer programming.

Standard Problem

Linear Program:

$$
\begin{aligned}
\max\ & c^{\mathrm{T}} x \\
\text{s.t. } & Ax = b \\
& x \geq 0
\end{aligned}
$$

Simplex Algorithm

History

- Simplex can be viewed as a combinatorial search for the optimal columns of the constraint matrix: columns that induce an optimal basis.
- There is an exponential number of column combinations.
- Despite this large number, Simplex is efficient in practice.

Sample Problem

Consider the following linear program:

$$\begin{array}{lll} \max & 5x_1 + 4x_2 + 3x_3 & \\ \text{s.t.}: & & \\ & 2x_1 + 3x_2 + x_3 & \leq \quad 5 \\ & 4x_1 + x_2 + 2x_3 & \leq \quad 11 \\ & 3x_1 + 4x_2 + 2x_3 & \leq \quad 8 \\ & x_1, x_2, x_3 \geq 0 & \end{array} \tag{1}$$

Sample Problem

After introducing slack variables, the problem assumes the form:

$$
\begin{array}{llcrcrcrcr}
\max & \delta & = & 0 & + & 5x_1 & + & 4x_2 & + & 3x_3 \\
s.t.: & & & & & & & & & \\
 & w_1 & = & 5 & - & 2x_1 & - & 3x_2 & - & x_3 \\
 & w_2 & = & 11 & - & 4x_1 & - & x_2 & - & 2x_3 \\
 & w_3 & = & 8 & - & 3x_1 & - & 4x_2 & - & 2x_3 \\
 & & & \multicolumn{7}{c}{x_1, x_2, x_3, w_1, w_2, w_3 \geq 0}
\end{array}
\tag{2}
$$

Sample Problem

Dictionary

- System (2) casts the LP problem in a form known as "dictionary."
- In the "dictionary", the solution is given in terms of a subset of variables (*basic variables*), with cardinality equal to the number of constraints, which are given as a function of the remaining variables (*nonbasic variables*).
- The nonbasic variables assume value "zero."
- Thus, a solution can be obtained directly from the dictionary.

Sample Problem

Dictionary

- System (2) casts the LP problem in a form known as "dictionary."
- In the "dictionary", the solution is given in terms of a subset of variables (*basic variables*), with cardinality equal to the number of constraints, which are given as a function of the remaining variables (*nonbasic variables*).
- The nonbasic variables assume value "zero."
- Thus, a solution can be obtained directly from the dictionary.

Sample Problem

Dictionary

- System (2) casts the LP problem in a form known as "dictionary."
- In the "dictionary", the solution is given in terms of a subset of variables (*basic variables*), with cardinality equal to the number of constraints, which are given as a function of the remaining variables (*nonbasic variables*).
- The nonbasic variables assume value "zero."
- Thus, a solution can be obtained directly from the dictionary.

Sample Problem

Dictionary (continued)

- In the above dictionary, the basis is formed by variables w_1, w_2 and w_3.
- The nonbasic variables are x_1, x_2 and x_3.
- Since nonbasic variables assume value zero, we obtain a solution from the basic variables: $w_1 = 5$, $w_2 = 11$ and $w_3 = 8$.
- The resulting solution is feasible:
  - All variables (basic and nonbasic) are nonnegative.
  - The objective function has value $\delta = 0$.

Sample Problem

Dictionary (continued)

- In the above dictionary, the basis is formed by variables w_1, w_2 and w_3.
- The nonbasic variables are x_1, x_2 and x_3.
- Since nonbasic variables assume value zero, we obtain a solution from the basic variables: $w_1 = 5$, $w_2 = 11$ and $w_3 = 8$.
- The resulting solution is feasible:
  - All variables (basic and nonbasic) are nonnegative.
  - The objective function has value $\delta = 0$.

Sample Problem

Dictionary (continued)

- In the above dictionary, the basis is formed by variables w_1, w_2 and w_3.
- The nonbasic variables are x_1, x_2 and x_3.
- Since nonbasic variables assume value zero, we obtain a solution from the basic variables: $w_1 = 5$, $w_2 = 11$ and $w_3 = 8$.
- The resulting solution is feasible:
  - All variables (basic and nonbasic) are nonnegative.
  - The objective function has value $\delta = 0$.

Principles of Simplex

Iterative method

- Simplex is an iterative process that starts with a solution $y^0 = (x_1^0 \ \dots \ x_n^0 \ w_1^0 \ \dots \ w_m^0)^T$, in which $n = 3$ and $m = 3$, satisfying the equations of (1).
- Starting from (2), Simplex seeks a new solution y^1 such that: $5x_1^1 + 4x_2^1 + 3x_3^1 > 5x_1^0 + 4x_2^0 + 3x_3^0$.
- To that end, it is necessary to bring a nonbasic variable, with a positive coefficient in the equation δ, into the basis.
- This will raise the value of the nonbasic variable entering the basis, thereby increasing the objective.

Principles of Simplex

Iterative method

- Simplex is an iterative process that starts with a solution $y^0 = (x_1^0 \ \ldots \ x_n^0 \ w_1^0 \ \ldots \ w_m^0)^T$, in which $n = 3$ and $m = 3$, satisfying the equations of (1).
- Starting from (2), Simplex seeks a new solution y^1 such that: $5x_1^1 + 4x_2^1 + 3x_3^1 > 5x_1^0 + 4x_2^0 + 3x_3^0$.
- To that end, it is necessary to bring a nonbasic variable, with a positive coefficient in the equation δ, into the basis.
- This will raise the value of the nonbasic variable entering the basis, thereby increasing the objective.

Principles of Simplex

Iterative method

- Simplex is an iterative process that starts with a solution $y^0 = (x_1^0 \ \ldots \ x_n^0 \ w_1^0 \ \ldots \ w_m^0)^T$, in which $n = 3$ and $m = 3$, satisfying the equations of (1).
- Starting from (2), Simplex seeks a new solution y^1 such that: $5x_1^1 + 4x_2^1 + 3x_3^1 > 5x_1^0 + 4x_2^0 + 3x_3^0$.
- To that end, it is necessary to bring a nonbasic variable, with a positive coefficient in the equation δ, into the basis.
- This will raise the value of the nonbasic variable entering the basis, thereby increasing the objective.

Principles of Simplex

Iterative method (continued)

- Obviously, the variable that enters the basis cannot increase indefinitely, unless the problem is unbounded.
- The first basic variable to become zero must leave the basis, becoming nonbasic at the next iteration.
- The process is repeated until convergence to the optimum, or the problem is detected to be unbounded.

Principles of Simplex

Iterative method (continued)

- Obviously, the variable that enters the basis cannot increase indefinitely, unless the problem is unbounded.
- The first basic variable to become zero must leave the basis, becoming nonbasic at the next iteration.
- The process is repeated until convergence to the optimum, or the problem is detected to be unbounded.

Simplex Algorithm: Example

Initialization

To start the iterative process, we need a feasible starting solution such as:

$$x_1^0 = 0,\ x_2^0 = 0,\ x_3^0 = 0,\ w_1^0 = 5,\ w_2^0 = 11,\ w_3^0 = 8$$

This solution y^0 induces an objective value $\delta = 0$.

Simplex Algorithm: Example

Step 1

- The incumbent solution is not optimal.
- Any increase in the value of x_1 increases the value of δ.
- The value of x_1 cannot increase arbitrarily since it is limited by the inequalities:

$$\begin{cases} w_1 = 5 - 2x_1 \geq 0 \\ w_2 = 11 - 4x_1 \geq 0 \\ w_3 = 8 - 3x_1 \geq 0 \end{cases} \Rightarrow \begin{cases} x_1 \leq 5/2 = 2.5 \\ x_1 \leq 11/4 = 2.75 \\ x_1 \leq 8/3 = 2.667 \end{cases}$$

Simplex Algorithm: Example

Step 1

- The incumbent solution is not optimal.
- Any increase in the value of x_1 increases the value of δ.
- The value of x_1 cannot increase arbitrarily since it is limited by the inequalities:

$$\left\{\begin{array}{lcrcl} w_1 & = & 5-2x_1 & \geq & 0 \\ w_2 & = & 11-4x_1 & \geq & 0 \\ w_3 & = & 8-3x_1 & \geq & 0 \end{array}\right. \Rightarrow \left\{\begin{array}{lclcr} x_1 & \leq & 5/2 & = & 2.5 \\ x_1 & \leq & 11/4 & = & 2.75 \\ x_1 & \leq & 8/3 & = & 2.667 \end{array}\right.$$

Simplex Algorithm: Example

Step 1 (continued)

Thus, the value of x_1 at the next iteration should be less than $\{5/2, 11/4, 8/3\}$, which leads to:

$$x_1 = \frac{5}{2} - \frac{1}{2}w_1 - \frac{3}{2}x_2 - \frac{1}{2}x_3 \tag{3}$$

Simplex Algorithm: Example

Step 1 (continued)

Replacing (3) in the system (2) in order to move w_1 to the right-hand side, we obtain:

$$
\begin{array}{rcl}
x_1 & = & \frac{5}{2} - \frac{1}{2}w_1 - \frac{3}{2}x_2 - \frac{1}{2}x_3 \\
 & & \\
w_2 & = & 11 - 4\left(\frac{5}{2} - \frac{1}{2}w_1 - \frac{3}{2}x_2 - \frac{1}{2}x_3\right) - x_2 - 2x_3 \\
 & = & 1 + 2w_1 + 5x_2 \\
 & & \\
w_3 & = & 8 - 3\left(\frac{5}{2} - \frac{1}{2}w_1 - \frac{3}{2}x_2 - \frac{1}{2}x_3\right) - 4x_2 - 2x_3 \\
 & = & \frac{1}{2} + \frac{3}{2}w_1 + \frac{1}{2}x_2 - \frac{1}{2}x_3 \\
 & & \\
\delta & = & 5\left(\frac{5}{2} - \frac{1}{2}w_1 - \frac{3}{2}x_2 - \frac{1}{2}x_3\right) + 4x_2 + 3x_3 \\
 & = & \frac{25}{2} - \frac{5}{2}w_1 - \frac{7}{2}x_2 + \frac{1}{2}x_3
\end{array}
\tag{4}
$$

Simplex Algorithm: Example

Step 1 (continued)

Now, replacing equations (4) in the "dictionary" (2), the following dictionary results:

$$
\begin{array}{lrcrrrr}
\max & \delta & = & \frac{25}{2} & -\frac{5}{2}w_1 & -\frac{7}{2}x_2 & +\frac{1}{2}x_3 \\
 & x_1 & = & \frac{5}{2} & -\frac{1}{2}w_1 & -\frac{3}{2}x_2 & -\frac{1}{2}x_3 \\
 & w_2 & = & 1 & +2w_1 & +5x_2 & \\
 & w_3 & = & \frac{1}{2} & +\frac{3}{2}w_1 & +\frac{1}{2}x_2 & -\frac{1}{2}x_3
\end{array}
\tag{5}
$$

Simplex Algorithm: Example

Step 1 (continued)

- The solution induced by the dictionary (5) is $y^1 = (x_1^1, x_2^1, x_3^1, w_1^1, w_2^1, w_3^1) = (\frac{5}{2}, 0, 0, 0, 1, \frac{1}{2})$ with objective é $\delta = \frac{25}{2}$.
- In this dictionary, the variables x_1, w_2, and w_3 are basic such that the set $\mathbb{B} = \{x_1, w_2, w_3\}$ contains all basic variables.
- The remaining variables $\mathbb{N} = \{x_2, x_3, w_1\}$ form the set of nonbasic variables.

Simplex Algorithm: Example

Step 1 (continued)

- The solution induced by the dictionary (5) is $y^1 = (x_1^1, x_2^1, x_3^1, w_1^1, w_2^1, w_3^1) = (\frac{5}{2}, 0, 0, 0, 1, \frac{1}{2})$ with objective é $\delta = \frac{25}{2}$.
- In this dictionary, the variables x_1, w_2, and w_3 are basic such that the set $\mathbb{B} = \{x_1, w_2, w_3\}$ contains all basic variables.
- The remaining variables $\mathbb{N} = \{x_2, x_3, w_1\}$ form the set of nonbasic variables.

Simplex Algorithm: Example

Step 2

- The incumbent solution is not optimal!
- Notice that a small increase in the value of x_3 will invariable increase the value of δ.
- We cannot increase the value of x_3 indefinitely because it will turn the solution infeasible — other variables may become negative.

Simplex Algorithm: Example

Step 2

- The incumbent solution is not optimal!
- Notice that a small increase in the value of x_3 will invariable increase the value of δ.
- We cannot increase the value of x_3 indefinitely because it will turn the solution infeasible — other variables may become negative.

Simplex Algorithm: Example

Step 2 (continued)

For the solution to remain feasible, the following inequalities must be respected:

$$\left\{\begin{array}{lcccc} x_1 & = & \frac{5}{2} - \frac{1}{2}x_3 & \geq & 0 \\ w_3 & = & \frac{1}{2} - \frac{1}{2}x_3 & \geq & 0 \end{array}\right. \Rightarrow \left\{\begin{array}{l} x_3 \leq 5 \\ x_3 \leq 1 \end{array}\right.$$

Simplex Algorithm: Example

Step 2 (continued)

- Therefore, w_3 must leave the basis so that x_3 enter the basis without violating any constraint.
- After replacing equation $x_3 = 1 + 3w_1 + x_2 - 2w_3$ in the equations of the dictionary (5), we obtain:

$$
\begin{aligned}
x_1 &= \tfrac{5}{2} - \tfrac{1}{2}w_1 - \tfrac{3}{2}x_2 - \tfrac{1}{2}(1 + 3w_1 + x_2 - 2w_3) \\
&= 2 - 2w_1 - 2x_2 + w_3 \\
\\
\delta &= \tfrac{25}{2} - \tfrac{5}{2}w_1 - \tfrac{7}{2}x_2 + \tfrac{1}{2}(1 + 3w_1 + x_2 - 2w_3) \\
&= 13 - w_1 - 3x_2 - w_3
\end{aligned}
\tag{6}
$$

Simplex Algorithm: Example

Step 2 (continued)

Replacing the equations of the dictionary (5) by the equations (6) results into a new dictionary:

$$
\begin{array}{lrcrrrr}
\text{Max} & \delta & = & 13 & -w_1 & -3x_2 & -w_3 \\
 & x_1 & = & 2 & -2w_1 & -2x_2 & +w_3 \\
 & w_2 & = & 1 & +2w_1 & +5x_2 & \\
 & x_3 & = & 1 & +3w_1 & +x_2 & -2w_3
\end{array} \tag{7}
$$

whose basis is $\mathbb{B} = \{x_1, x_3, w_2\}$.

Simplex Algorithm: Example

Step 2 (continued)

The solution given by the dictionary (7) is optimal:

- $x_1 = 2$, $x_2 = 0$, $x_3 = 1$, $w_1 = 0$, $w_2 = 1$, $w_3 = 0$
- $\delta = 13$ is the optimal value because the coefficients of the nonbasic variables in the equation δ are all negative, in the dictionary given by (7),
- Increasing the value of any nonbasic variable will reduce the value of the objective.

Simplex Algorithm: Example

Step 2 (continued)

The solution given by the dictionary (7) is optimal:

- $x_1 = 2$, $x_2 = 0$, $x_3 = 1$, $w_1 = 0$, $w_2 = 1$, $w_3 = 0$
- $\delta = 13$ is the optimal value because the coefficients of the nonbasic variables in the equation δ are all negative, in the dictionary given by (7),
- Increasing the value of any nonbasic variable will reduce the value of the objective.

Summary

Introduction

Simplex Algorithm

Initial Solution

Duality

Auxiliary Problem

Original Problem

$$
\begin{array}{ll}
\text{Maximize} & -2x_1 - x_2 \\
\text{Subject to :} & \\
& \begin{array}{rcrcr}
-x_1 & + & x_2 & \leq & -1 \\
-x_1 & - & 2x_2 & \leq & -2 \\
 & & x_2 & \leq & 1
\end{array} \\
& x_1, x_2 \geq 0
\end{array}
$$

Auxiliary Problem

Auxiliary Problem

$$
\begin{array}{llllllll}
\max & -x_0 & & & & & & \\
s.t.: & & & & & & & \\
& -x_1 & + & x_2 & - & x_0 & \leq & -1 \\
& -x_1 & - & 2x_2 & - & x_0 & \leq & -2 \\
& & & x_2 & - & x_0 & \leq & 1 \\
& x_0, x_1, x_2 \geq 0 & & & & & &
\end{array}
$$

Auxiliary Problem

Initial Dictionary

$$
\begin{array}{rrcrrrr}
\max & \delta & = & & & & -x_0 \\
 & w_1 & = & -1 & +x_1 & -x_2 & +x_0 \\
 & w_2 & = & -2 & +x_1 & +2x_2 & +x_0 \\
 & w_3 & = & 1 & & -x_2 & +x_0
\end{array}
$$

Summary

Introduction

Simplex Algorithm

Initial Solution

Duality

Duality

- Associated to any linear programming problem P (primal) is another linear programming problem, the dual, which is denoted D.
- Theoretical consequences.
- Any solution for the dual D induces a limit for the optimum value of the primal P, and vice versa.

Duality

- Associated to any linear programming problem P (primal) is another linear programming problem, the dual, which is denoted D.
- Theoretical consequences.
- Any solution for the dual D induces a limit for the optimum value of the primal P, and vice versa.

Motivation

Consider the sample problem:

$$
\begin{array}{lllll}
P: & \max & 4x_1 + x_2 + 3x_3 & & \\
 & \text{s.t.}: & & & \\
 & & x_1 + 4x_2 & \leq & 1 \\
 & & 3x_1 - x_2 + x_3 & \leq & 3 \\
 & & x_1, x_2, x_3 & \geq & 0
\end{array}
\tag{8}
$$

Any feasible solution to P induces a lower bound.

- For example, $x' = (1, 0, 0)$ shows that the optimum objective $\delta^\star \geq 4$.
- Using the solution $x'' = (0, 0, 3)$, we discover that $\delta^\star \geq 9$.

Motivation

Consider the sample problem:

$$
\begin{array}{lll}
P: & \max & 4x_1 + x_2 + 3x_3 \\
& \text{s.t.}: & \\
& & x_1 + 4x_2 \quad \leq \quad 1 \\
& & 3x_1 - x_2 + x_3 \quad \leq \quad 3 \\
& & x_1, x_2, x_3 \quad \geq \quad 0
\end{array}
\tag{8}
$$

Any feasible solution to P induces a lower bound.

- For example, $x' = (1, 0, 0)$ shows that the optimum objective $\delta^\star \geq 4$.
- Using the solution $x'' = (0, 0, 3)$, we discover that $\delta^\star \geq 9$.

Motivation

Issues

- Are these lower bounds close to the optimum?
- Let's multiply the first constraint of (8) by 2, multiply the second constraint by 3, and then add them up as follows:

$$\begin{array}{rcl} 2(x_1 + 4x_2) & \leq & 2(1) \\ 3(3x_1 - x_2 + x_3) & \leq & 3(3) \\ \hline 11x_1 + 5x_2 + 3x_3 & \leq & 11 \end{array}$$

- Notice that $4x_1 + x_2 + 3x_3 \leq 11x_1 + 5x_2 + 3x_3 \leq 11$, because $x_j \geq 0$.
- Then we conclude that $9 \leq \delta^\star \leq 11$.

Motivation

Issues

- Are these lower bounds close to the optimum?
- Let's multiply the first constraint of (8) by 2, multiply the second constraint by 3, and then add them up as follows:

$$\begin{array}{rcl} 2(x_1 + 4x_2) & \leq & 2(1) \\ 3(3x_1 - x_2 + x_3) & \leq & 3(3) \\ \hline 11x_1 + 5x_2 + 3x_3 & \leq & 11 \end{array}$$

- Notice that $4x_1 + x_2 + 3x_3 \leq 11x_1 + 5x_2 + 3x_3 \leq 11$, because $x_j \geq 0$.
- Then we conclude that $9 \leq \delta^\star \leq 11$.

Motivation

Issues

- Are these lower bounds close to the optimum?
- Let's multiply the first constraint of (8) by 2, multiply the second constraint by 3, and then add them up as follows:

$$\begin{array}{lcl} 2(x_1 + 4x_2) & \leq & 2(1) \\ 3(3x_1 - x_2 + x_3) & \leq & 3(3) \\ \hline 11x_1 + 5x_2 + 3x_3 & \leq & 11 \end{array}$$

- Notice that $4x_1 + x_2 + 3x_3 \leq 11x_1 + 5x_2 + 3x_3 \leq 11$, because $x_j \geq 0$.
- Then we conclude that $9 \leq \delta^\star \leq 11$.

Motivation

- To obtain tight bound, we follow the same procedure but, this time, we use variables rather than fixed values.
- By multiplying the constraints with nonnegative variables:

$$
\begin{array}{rcl}
y_1(x_1 + 4x_2) & \leq & y_1 \\
y_2(3x_1 - x_2 + x_3) & \leq & 3y_2 \\
\hline
(y_1 + 3y_2)x_1 + (4y_1 - y_2)x_2 + y_2x_3 & \leq & y_1 + 3y_2
\end{array}
$$

Motivation

- To obtain tight bound, we follow the same procedure but, this time, we use variables rather than fixed values.
- By multiplying the constraints with nonnegative variables:

$$
\begin{array}{lcl}
y_1(x_1 + 4x_2) & \leq & y_1 \\
y_2(3x_1 - x_2 + x_3) & \leq & 3y_2 \\
\hline
(y_1 + 3y_2)x_1 + (4y_1 - y_2)x_2 + y_2x_3 & \leq & y_1 + 3y_2
\end{array}
$$

Motivation

Now, we stipulate that:

$$\begin{array}{lcl} y_1 + 3y_2 & \geq & 4 \\ 4y_1 - y_2 & \geq & 1 \\ y_2 & \geq & 3 \\ y_1, y_2 & \geq & 0 \end{array} \tag{9}$$

Values for (y_1, y_2) satisfying the inequalities (9) lead to the following inequalities:

$$\begin{array}{lcl} \delta & = & 4x_1 + x_2 + 3x_3 \\ & \leq & (y_1 + 3y_2)x_1 + (4y_1 - y_2)x_2 + y_2x_3 \\ & \leq & y_1 + 3y_2 \end{array}$$

Motivation

Now, we stipulate that:

$$\begin{array}{lcl} y_1 + 3y_2 & \geq & 4 \\ 4y_1 - y_2 & \geq & 1 \\ y_2 & \geq & 3 \\ y_1, y_2 & \geq & 0 \end{array} \tag{9}$$

Values for (y_1, y_2) satisfying the inequalities (9) lead to the following inequalities:

$$\begin{array}{rcl} \delta & = & 4x_1 + x_2 + 3x_3 \\ & \leq & (y_1 + 3y_2)x_1 + (4y_1 - y_2)x_2 + y_2 x_3 \\ & \leq & y_1 + 3y_2 \end{array}$$

Motivation

- We obtain an upper bound $(y_1 + 3y_2)$ for $\delta^\star$.
- We wish to minimize this upper bound:

$$
\begin{array}{rlrcl}
D: & \min & y_1 + 3y_2 & & \\
& \text{s.t.}: & & & \\
& & y_1 + 3y_2 & \geq & 4 \\
& & 4y_1 - y_2 & \geq & 1 \\
& & y_2 & \geq & 3 \\
& & y_1, y_2 & \geq & 0
\end{array}
$$

Motivation

- We obtain an upper bound $(y_1 + 3y_2)$ for $\delta^\star$.
- We wish to minimize this upper bound:

$$
\begin{array}{lllll}
D: & \min & y_1 + 3y_2 & & \\
 & \text{s.t.}: & & & \\
 & & y_1 + 3y_2 & \geq & 4 \\
 & & 4y_1 - y_2 & \geq & 1 \\
 & & y_2 & \geq & 3 \\
 & & y_1, y_2 & \geq & 0
\end{array}
$$

Linear Programming

- Thank you for attending this lecture!!!