Piecewise-Linear Approximation: Multidimensional

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Motivation

CC Model

DCC Model

SOS2 Model

Summary

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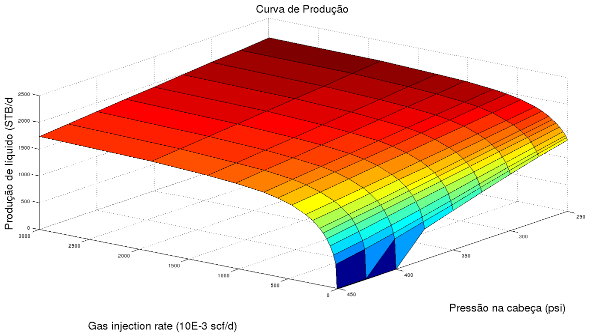
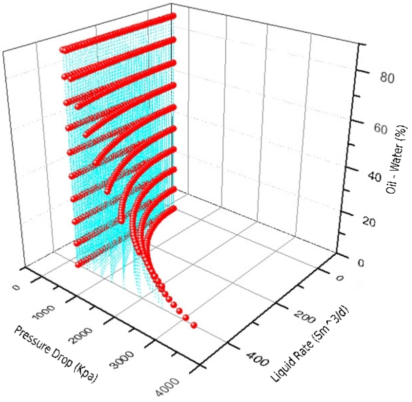
DCC Model

SOS2 Model

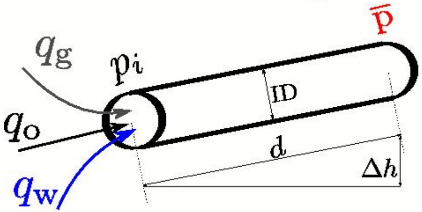
Well Production $\times$ (Lift-gas, Well-head Pressure).



Pressure Loss in Flowlines

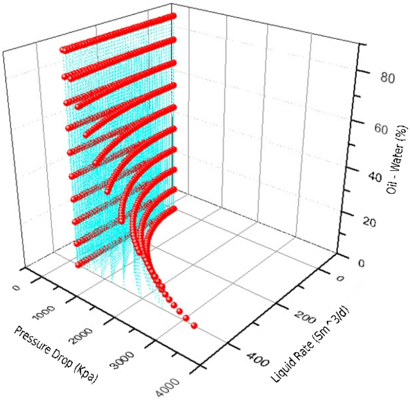


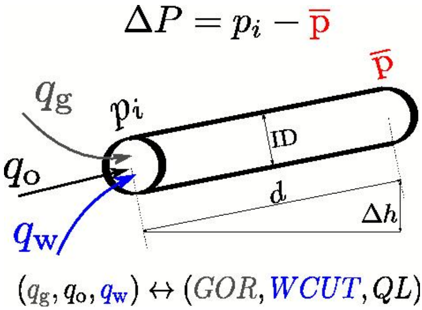
$$\Delta P = p_i - \bar{p}$$



$$(q_g, q_o, q_w) \leftrightarrow (GOR, WCUT, QL)$$

Pressure Loss in Flowlines





Multidimensional Models

Let us consider functions with multidimensional domains: $f : \mathcal{D} \to \mathbb{R}$, in which $\mathcal{D} \subset \mathbb{R}^n$ is the domain.

An issue that comes up is how to partition the domain.

- Let $\mathcal{P}$ be a set of polyhedrons defining a partitioning of $\mathcal{D}$:

$$\cup_{P \in \mathcal{P}} P = \mathcal{D},$$
$$\mathrm{int}(P) \cap P' = \emptyset, \forall P, P' \in \mathcal{P}$$

- The piecewise-linear models can be applied depending on the structure of the partitioning $\mathcal{P}$ (geometry/topology).

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Models and Partitioning

These models can be applied to any convex partitioning of the domain:

- CC, DCC, DLog e Multiple Choice.

Can be applied only to a J-1 triangulation of the domain:

- Log.

Can be applied only to regular lattices:

- SOS2.

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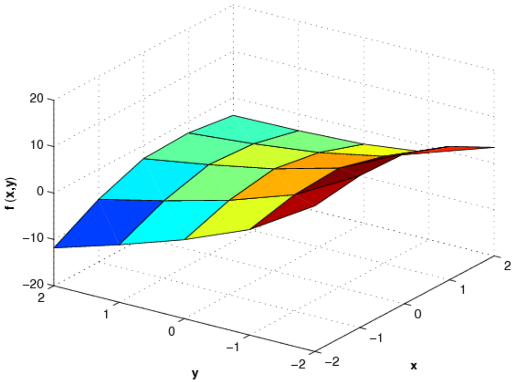

Summary

Motivation

CC Model

DCC Model

SOS2 Model

CC Model

Definitions:

- $\mathcal{P}$ is the set of polyhedra defining the domain.
- $\mathcal{V}$ is the set of breakpoints.
- $V(P) \subset \mathcal{V}$ is the set of vertices of polyhedron $P \in \mathcal{P}$.

Variables:

- $\lambda_v \in [0, 1]$ is a weighting variable associated with vertex $v \in \mathcal{V}$.
- $y_P \in \{0, 1\}$ assumes value 1 if $x \in P$, $P \in \mathcal{P}$, otherwise assumes value 0.

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CC Model

$$f = \sum_{v \in \mathcal{V}} f(v)\lambda_v, \qquad x = \sum_{v \in \mathcal{V}} v\lambda_v,$$

$$1 = \sum_{v \in \mathcal{V}} \lambda_v, \qquad 1 = \sum_{P \in \mathcal{P}} y_P,$$

$$\lambda_v \le \sum_{P \in \mathcal{P}(v)} y_P,$$

$$\lambda_v \ge 0,\ v \in \mathcal{V},$$

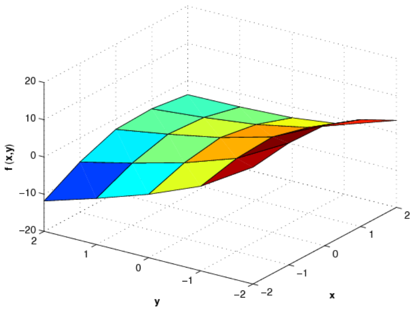
$$y_P \in \{0, 1\},\ P \in \mathcal{P}$$

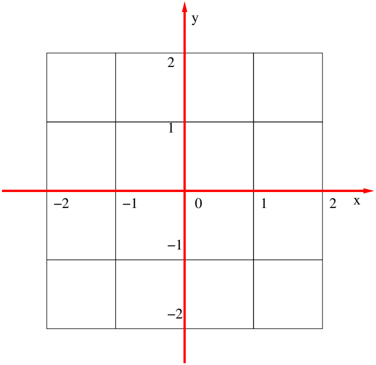
in which: $\mathcal{P}(v) = \{P \in \mathcal{P} : v \in V(P)\}$ is the set of polyhedra that contain breakpoint v.

Sample Function

Consider the nonlinear function:

$$f(x,y) = \mathrm{e}^{-y} + \mathrm{e}^{-x^2} - x^2 + xy - 2y$$





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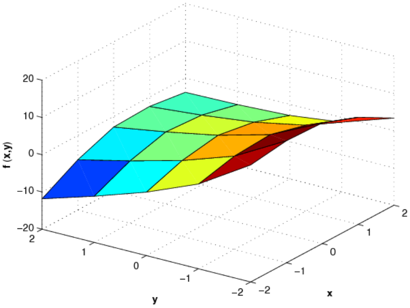

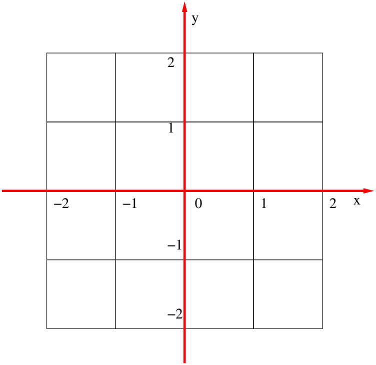

Example: CC Model

Vertex set:

$$\mathcal{V} = \{-2, -1, 0, 1, 2\} \times \{-2, -1, 0, 1, 2\}\}$$

Polyhedra set:

$$\mathcal{P} = \{-2, -1, 0, 1\} \times \{-2, -1, 0, 1\}\}$$

in which, the index $(i, j) \in \mathcal{P}$ corresponds to the polyhedron

$$P_{i,j} = \{(x, y) \in \mathbb{R}^2 : i \leq x \leq i + 1, j \leq y \leq j + 1\}$$

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$$P_{i,j} = \{(x, y) \in \mathbb{R}^2 : i \leq x \leq i + 1, j \leq y \leq j + 1\}$$

Example: CC Model

$$
\begin{aligned}
f &= \sum_{v \in \mathcal{V}} f(v)\lambda_v \\
&= f(-2,-2)\lambda_{(-2,-2)} + f(-2,-1)\lambda_{(-2,-1)} + f(-2,0)\lambda_{(-2,0)} \\
&\qquad + f(-2,1)\lambda_{(-2,1)} + f(-2,2)\lambda_{(-2,2)} \\
&\qquad \vdots \\
&\quad + f(2,-2)\lambda_{(2,-2)} + f(2,-1)\lambda_{(2,-1)} + f(2,0)\lambda_{(2,0)} \\
&\qquad + f(2,1)\lambda_{(2,1)} + f(2,2)\lambda_{(2,2)}
\end{aligned}
$$

Example: CC Model

$$
\begin{aligned}
x &= \sum_{v \in \mathcal{V}} v_x \lambda_v \\
&= (-2)\lambda_{(-2,-2)} + (-2)\lambda_{(-2,-1)} + (-2)\lambda_{(-2,0)} + (-2)\lambda_{(-2,1)} + (-2)\lambda_{(-2,2)} \\
&\quad \vdots \\
&\quad + (2)\lambda_{(2,-2)} + (2)\lambda_{(2,-1)} + (2)\lambda_{(2,0)} + (2)\lambda_{(2,1)} + (2)\lambda_{(2,2)}
\end{aligned}
$$

Example: CC Model

$$
\begin{aligned}
y &= \sum_{v \in \mathcal{V}} v_y \lambda_v \\
&= \quad (-2)\lambda_{(-2,-2)} + (-1)\lambda_{(-2,-1)} + (0)\lambda_{(-2,0)} + (1)\lambda_{(-2,1)} + (2)\lambda_{(-2,2)} \\
&\qquad \vdots \\
&\quad + (-2)\lambda_{(2,-2)} + (-1)\lambda_{(2,-1)} + (0)\lambda_{(2,0)} + (1)\lambda_{(2,1)} + (2)\lambda_{(2,2)}
\end{aligned}
$$

Example: CC Model

$$
\begin{aligned}
1 &= \sum_{v \in \mathcal{V}} \lambda_v \\
&= \lambda_{(-2,-2)} + \lambda_{(-2,-1)} + \lambda_{(-2,0)} + \lambda_{(-2,1)} + \lambda_{(-2,2)} \\
&\quad + \lambda_{(-1,-2)} + \lambda_{(-1,-1)} + \lambda_{(-1,0)} + \lambda_{(-1,1)} + \lambda_{(-1,2)} \\
&\qquad \vdots \\
&\quad + \lambda_{(2,-2)} + \lambda_{(2,-1)} + \lambda_{(2,0)} + \lambda_{(2,1)} + \lambda_{(2,2)}
\end{aligned}
$$

Example: CC Model

$$
\begin{aligned}
1 &= \sum_{P \in \mathcal{P}} y_P \\
&= y_{(-2,-2)} + y_{(-2,-1)} + y_{(-2,0)} + y_{(-2,1)} \\
&\quad + y_{(-1,-2)} + y_{(-1,-1)} + y_{(-1,0)} + y_{(-1,1)} \\
&\qquad \vdots \\
&\quad + y_{(2,-2)} + y_{(2,-1)} + y_{(2,0)} + y_{(2,1)}
\end{aligned}
$$

Example: CC Model

The constraint family

$$\lambda_v \leq \sum_{P \in \mathcal{P}(v)} y_P,$$

becomes

$$\begin{aligned}
\lambda_{(-2,-2)} &\leq y_{(-2,-2)} \\
\lambda_{(-2,-1)} &\leq y_{(-2,-2)} + y_{(-2,-1)} \\
\lambda_{(-2,0)} &\leq y_{(-2,-1)} + y_{(-2,0)} \\
\lambda_{(-2,1)} &\leq y_{(-2,0)} + y_{(-2,1)} \\
\lambda_{(-2,2)} &\leq y_{(-2,1)} \\
\vdots \quad & \quad \vdots
\end{aligned}
\qquad
\begin{aligned}
\lambda_{(-1,-2)} &\leq y_{(-2,-2)} + y_{(-2,-1)} \\
\lambda_{(-1,-1)} &\leq y_{(-2,-2)} + y_{(-2,-1)} + y_{-1,-2} + y_{(-1,-1)} \\
\lambda_{(-1,0)} &\leq y_{(-2,-1)} + y_{(-1,-1)} + y_{(-2,0)} + y_{(-1,0)} \\
\lambda_{(-1,1)} &\leq y_{(-2,0)} + y_{(-1,0)} + y_{(-2,1)} + y_{(-1,1)} \\
\lambda_{(-1,2)} &\leq y_{(-2,1)} + y_{(-1,1)} \\
\vdots \quad & \quad \vdots
\end{aligned}$$

Summary

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SOS2 Model

Disaggregated Convex Combination (DCC) Model

$$f = \sum_{P \in \mathcal{P}} \sum_{v \in \mathcal{V}(P)} f(v)\lambda_{P,v}, \qquad x = \sum_{P \in \mathcal{P}} \sum_{v \in \mathcal{V}(P)} v\lambda_{P,v},$$

$$y_P = \sum_{v \in \mathcal{V}(P)} \lambda_{P,v}, \ P \in \mathcal{P},$$

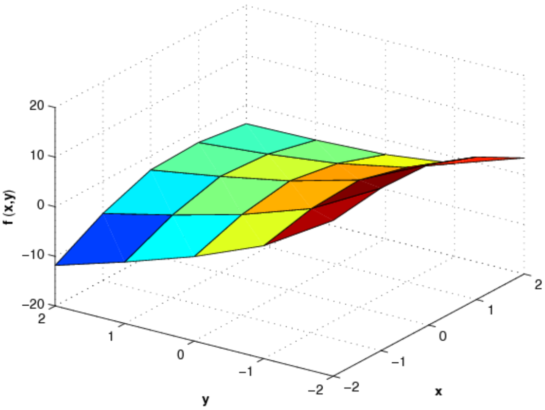
$$1 = \sum_{P \in \mathcal{P}} y_P,$$

$$\lambda_{P,v} \geq 0, \ P \in \mathcal{P}, \ v \in \mathcal{V}(P),$$

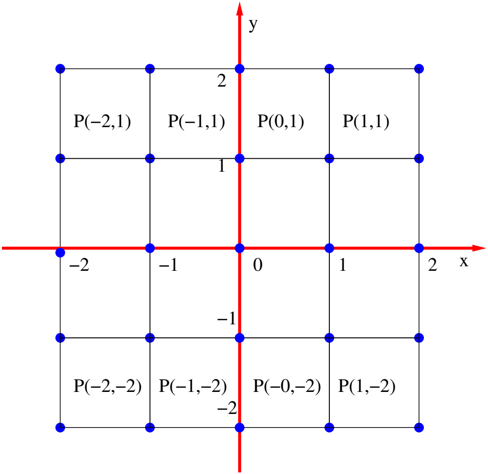
$$y_P \in \{0,1\}, \ P \in \mathcal{P}$$

where: $\lambda_{P,v}$ is associated to vertex v of polyhedron P (breakpoint).

Example: Function



Example: Domain



Example: DCC Model

$$
\begin{aligned}
f &= \sum_{P \in \mathcal{P}} \sum_{v \in \mathcal{V}(P)} f(v)\lambda_{P,v}, \\
&= \quad f(-2,-2)\lambda_{(-2,-2)}^{P(-2,-2)} + f(-2,-1)\lambda_{(-2,-1)}^{P(-2,-2)} \\
&\qquad + f(-1,-2)\lambda_{(-1,-2)}^{P(-2,-2)} + f(-1,-1)\lambda_{(-1,-1)}^{P(-2,-2)} \\
&\quad + f(-1,-2)\lambda_{(-1,-2)}^{P(-1,-2)} + f(-1,-1)\lambda_{(-1,-1)}^{P(-1,-2)} \\
&\qquad + f(0,-2)\lambda_{(0,-2)}^{P(-1,-2)} + f(0,-1)\lambda_{(0,-1)}^{P(-1,-2)} \\
&\qquad \vdots \\
&\quad + f(1,1)\lambda_{(1,1)}^{P(1,1)} + f(1,2)\lambda_{(1,2)}^{P(1,1)} \\
&\qquad + f(2,1)\lambda_{(2,1)}^{P(1,1)} + f(2,2)\lambda_{(2,2)}^{P(1,1)}
\end{aligned}
$$

Example: DCC Model

$$
\begin{aligned}
(x,y) &= \sum_{P\in\mathcal{P}} \sum_{v\in\mathcal{V}(P)} v\lambda_{P,v}, \\
&= \quad (-2,-2)\lambda^{P(-2,-2)}_{(-2,-2)} + (-2,-1)\lambda^{P(-2,-2)}_{(-2,-1)} \\
&\qquad +(-1,-2)\lambda^{P(-2,-2)}_{(-1,-2)} + (-1,-1)\lambda^{P(-2,-2)}_{(-1,-1)} \\
&\quad +(-1,-2)\lambda^{P(-1,-2)}_{(-1,-2)} + (-1,-1)\lambda^{P(-1,-2)}_{(-1,-1)} \\
&\qquad +(0,-2)\lambda^{P(-1,-2)}_{(0,-2)} + (0,-1)\lambda^{P(-1,-2)}_{(0,-1)} \\
&\qquad \vdots \\
&\quad +(1,1)\lambda^{P(1,1)}_{(1,1)} + (1,2)\lambda^{P(1,1)}_{(1,2)} \\
&\qquad +(2,1)\lambda^{P(1,1)}_{(2,1)} + (2,2)\lambda^{P(1,1)}_{(2,2)}
\end{aligned}
$$

Example: DCC Model

The constraint family

$$y_P = \sum_{v \in \mathcal{V}(P)} \lambda_{P,v}, \; P \in \mathcal{P},$$

becomes

$$\begin{aligned}
y_{P(-2,-2)} &= \lambda^{P(-2,-2)}_{(-2,-2)} + \lambda^{P(-2,-2)}_{(-2,-1)} + \lambda^{P(-2,-2)}_{(-1,-2)} + \lambda^{P(-2,-2)}_{(-1,-1)} \\
y_{P(-1,-2)} &= \lambda^{P(-1,-2)}_{(-1,-2)} + \lambda^{P(-1,-2)}_{(-1,-1)} + \lambda^{P(-1,-2)}_{(0,-2)} + \lambda^{P(-1,-2)}_{(0,-1)} \\
\vdots \quad & \quad \vdots \\
y_{P(1,1)} &= \lambda^{P(1,1)}_{(1,1)} + \lambda^{P(1,1)}_{(1,2)} + \lambda^{P(1,1)}_{(2,1)} + \lambda^{P(1,1)}_{(2,2)}
\end{aligned}$$

Example: DCC Model

The constraint family

$$1 = \sum_{P \in \mathcal{P}} y_P,$$

becomes

$$1 = y_{P(-2,-2)} + y_{P(-1,-2)} + y_{P(0,-2)} + \ldots + y_{P(1,1)}$$

Summary

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CC Model

DCC Model

SOS2 Model

SOS2 Model

Objectives:

- Piecewise-linear approximation of the function $f : \mathbb{R}^2 \to \mathbb{R}$.

Definitions:

- Breakpoint set in $x : \mathcal{X} = \{x_1, x_2, \ldots, x_n\}$.
- Breakpoint set in $y : \mathcal{Y} = \{y_1, y_2, \ldots, y_n\}$.
- Given the values of the function f for $(x, y) \in \mathcal{X} \times \mathcal{Y}$.
- Polyhedron set given by:

$$\mathcal{P} = \{[x_i, x_{i+1}] \times [y_j, y_{j+1}] : i = 1, \ldots, n-1, j = 1, \ldots, n-1\}$$

.

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.

SOS2 Model

$$f = \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} f(x, y)\lambda_{x,y},$$

$$x = \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} x\lambda_{x,y},$$

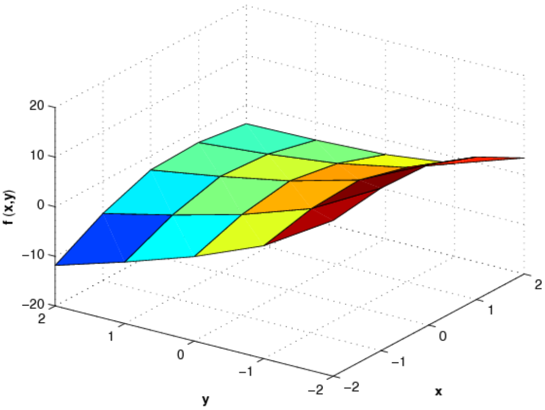
$$y = \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} y\lambda_{x,y},$$

$$1 = \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} \lambda_{x,y}, \qquad \lambda_{x,y} \geq 0, \ \forall x \in \mathcal{X}, y \in \mathcal{Y}$$

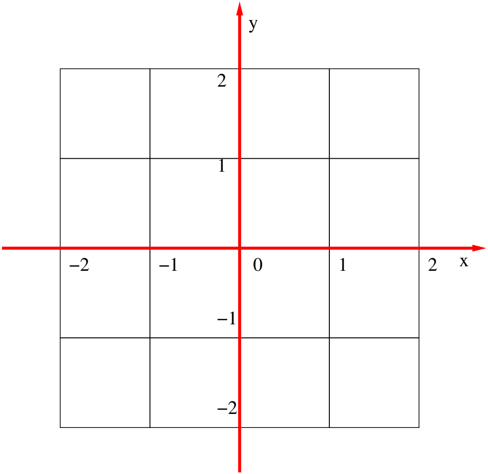
$$\eta_x = \sum_{y \in \mathcal{Y}} \lambda_{x,y}, \ \forall x \in \mathcal{X}, \qquad \eta_y = \sum_{x \in \mathcal{X}} \lambda_{x,y}, \ \forall y \in \mathcal{Y},$$

$(\eta_x)_{x \in \mathcal{X}}$ is SOS2 $\qquad (\eta_y)_{y \in \mathcal{Y}}$ is SOS2

Example: Function



Example: Domain



Example: SOS2 Model

Function approximation becomes

$$
\begin{aligned}
f &= \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} f(x,y)\lambda_{x,y} \\
&= +f(-2,-2)\lambda_{(-2,-2)} + f(-2,-1)\lambda_{(-2,-1)} + f(-2,0)\lambda_{(-2,0)} \\
&\quad +f(-2,1)\lambda_{(-2,1)} + f(-2,2)\lambda_{(-2,2)} \\
&\qquad\qquad \vdots \\
&\quad +f(2,-2)\lambda_{(2,-2)} + f(2,-1)\lambda_{(2,-1)} + f(2,0)\lambda_{(2,0)} \\
&\quad +f(2,1)\lambda_{(2,1)} + f(2,2)\lambda_{(2,2)}
\end{aligned}
$$

Example: SOS2 Model

Domain variables

$$
\begin{aligned}
(x, y) &= \sum_{\mathrm{x} \in \mathcal{X}} \sum_{\mathrm{y} \in \mathcal{Y}} (\mathrm{x}, \mathrm{y}) \lambda_{\mathrm{x},\mathrm{y}} \\
&= +(-2,-2)\lambda_{(-2,-2)} + (-2,-1)\lambda_{(-2,-1)} + (-2,0)\lambda_{(-2,0)} \\
&\quad +(-2,1)\lambda_{(-2,1)} + (-2,2)\lambda_{(-2,2)} \\
&\qquad \vdots \\
&\quad +(2,-2)\lambda_{(2,-2)} + (2,-1)\lambda_{(2,-1)} + (2,0)\lambda_{(2,0)} \\
&\quad +(2,1)\lambda_{(2,1)} + (2,2)\lambda_{(2,2)}
\end{aligned}
$$

Example: SOS2 Model

$$
\begin{aligned}
1 &= \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} \lambda_{x,y} \\
&= \lambda_{(-2,-2)} + \lambda_{(-2,-1)} + \lambda_{(-2,0)} + \lambda_{(-2,1)} + \lambda_{(-2,2)} \\
&\quad \vdots \\
&\quad + \lambda_{(2,-2)} + \lambda_{(2,-1)} + \lambda_{(2,0)} + \lambda_{(2,1)} + \lambda_{(2,2)}
\end{aligned}
$$

SOS2 Model

The constraint family

$$\eta_{\mathrm{x}} = \sum_{\mathrm{y} \in \mathcal{Y}} \lambda_{\mathrm{x},\mathrm{y}}, \; x \in \mathcal{X}$$

becomes

$$\begin{aligned}
\eta^{\mathrm{x}}_{-2} &= \lambda_{(-2,-2)} + \lambda_{(-2,-1)} + \lambda_{(-2,0)} + \lambda_{(-2,1)} + \lambda_{(-2,2)} \\
\eta^{\mathrm{x}}_{-1} &= \lambda_{(-1,-2)} + \lambda_{(-1,-1)} + \lambda_{(-1,0)} + \lambda_{(-1,1)} + \lambda_{(-1,2)} \\
\vdots &= \qquad\qquad \vdots \\
\eta^{\mathrm{x}}_{2} &= \lambda_{(2,-2)} + \lambda_{(2,-1)} + \lambda_{(2,0)} + \lambda_{(2,1)} + \lambda_{(2,2)}
\end{aligned}$$

SOS2 Model

The constraint family

$$\eta_{\mathrm{y}} = \sum_{\mathrm{x} \in \mathcal{X}} \lambda_{\mathrm{x},\mathrm{y}},\ y \in \mathcal{Y}$$

becomes

$$\begin{aligned}
\eta^{\mathrm{y}}_{-2} &= \lambda_{(-2,-2)} + \lambda_{(-1,-2)} + \lambda_{(0,-2)} + \lambda_{(1,-2)} + \lambda_{(2,-2)} \\
\eta^{\mathrm{y}}_{-1} &= \lambda_{(-2,-1)} + \lambda_{(-1,-1)} + \lambda_{(0,-1)} + \lambda_{(1,-1)} + \lambda_{(2,-1)} \\
\vdots &= \qquad\quad \vdots \\
\eta^{\mathrm{y}}_{2} &= \lambda_{(-2,2)} + \lambda_{(-1,2)} + \lambda_{(0,2)} + \lambda_{(1,2)} + \lambda_{(2,2)}
\end{aligned}$$

Piecewise-Linear Approximation: Multidimensional

- End!
- Thank you for your attention.