Piecewise-Linear Approximation: One Dimensional

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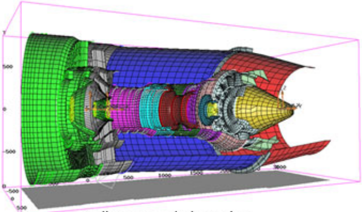
Summary

Surrogate Model

Engineering problems involve experiments, and simulations, to evaluate objectives and constraints which are functions of several variables.

Optimal design of jet engine.



Rolls-Royce whole engine structural model (VIVACE project)

Surrogate Model

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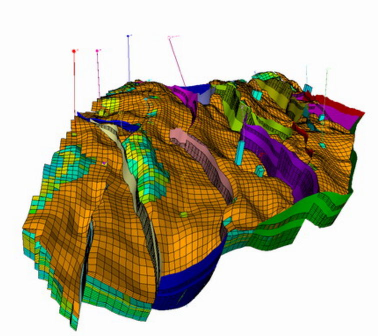
Advanced recovery of oil and gas reservoirs.



Surrogate Model

Challenges:

- In real-world problems, a simulation run can take several minutes, hours and even days.
- Design optimization and case studies may become impractical due to the potential need of thousands of simulations.

Surrogate Model

Alternative:

- Propose approximate models ("Surrogate Models") that emulate the behavior of systems and simulators, however at a low computational cost.
- Surrogate Models are built from data, since the simulation model might not be known or is too complicated to be expressed in explicit form.
- Knowledge of the input-output behavior is known or given.

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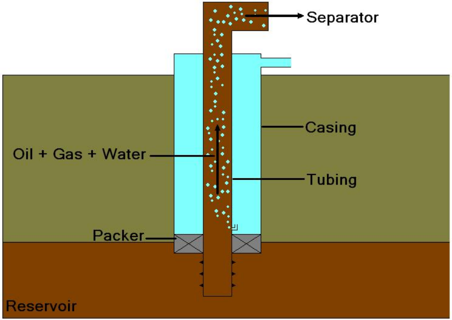
Surrogate Model

"All models are wrong, the practical question is how wrong do they have to be to not be useful." George Box

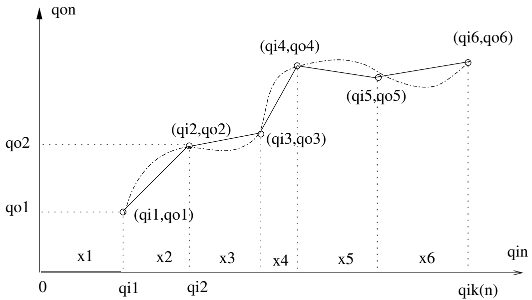
Piecewise-Linear Approximation

Models are obtained from linear (affine) combination of input-output data: $q_{oil}(q_{inj})$.



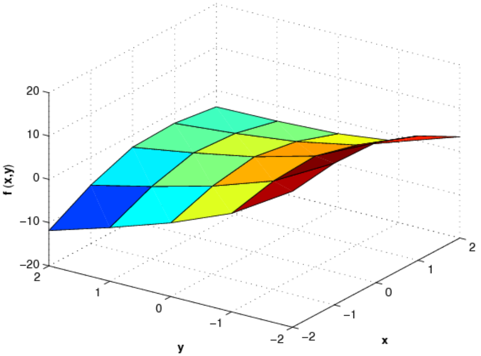
Piecewise-Linear Approximation

- Models are obtained from linear (affine) combination of input-output data: $q_{oil}(q_{inj})$.
- Data: $\{(q^1_{inj}, q^1_{oil}), (q^2_{inj}, q^2_{oil}), \ldots, (q^n_{inj}, q^n_{oil})\}$.



Piecewise-Linear Approximation

Function $f(x, y)$ with a two dimensional domain.



Piecewise-Linear Approximation

Question:

- How does one represent piecewise-linear functions in mathematical programming?

Several Models:

- CC (Convex Combination)
- Inc (Incremental)
- DCC (Disaggregated Convex Combination)
- Log (Logarithmic Convex Combination)
- DLog (Disaggregated Logarithmic Convex Combination)
- Multiple Choice
- SOS2 (Specially Ordered Sets of Type 2)

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Summary

Motivation

Piecewise-Linear Models

Convex Combination (CC)

Data: $\{(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)\}$.

$$x = \sum_{i=0}^{n} \lambda_i x_i \qquad y = \sum_{i=0}^{n} \lambda_i y_i$$

$$1 = \sum_{i=0}^{n} \lambda_i \qquad \lambda_i \geq 0,\ i = 0, \ldots, n$$

$$1 = \sum_{i=1}^{n} z_i \qquad z_i \in \{0, 1\},\ i = 1, \ldots, n$$

$$\lambda_0 \leq z_1 \qquad \lambda_i \leq z_i + z_{i+1},\ i = 1, \ldots, n-1$$

$$\lambda_n \leq z_n$$

Remark: $z_i = 1$ if $x \in [x_{i-1}, x_i]$.

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Incremental (INC)

Givens: $\{(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)\}$.

$$x = x_0 + \sum_{i=1}^{n} \delta_i \qquad\qquad y = y_0 + \sum_{i=1}^{n} \frac{(y_i - y_{i-1})}{(x_i - x_{i-1})}\delta_i$$

$$\delta_1 \leq (x_1 - x_0)$$

$$\delta_i \leq (x_i - x_{i-1})z_{i-1},\ i = 2, \ldots, n \qquad \delta_n \geq 0$$

$$\delta_i \geq (x_i - x_{i-1})z_i,\ i = 1, \ldots, n-1 \qquad z_i \in \{0, 1\},\ i = 1, \ldots, n-1$$

Remarks:

- If $z_i = 1$, then $z_j = 1$ for $j = 1, \ldots, i-1$.
- If $z_i = 1$, then $\delta_i = (x_i - x_{i-1})$.

Disaggregated Convex Combination (DCC)

Givens: $\{(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)\}$.

$$x = \sum_{i=1}^{n} (\lambda_i^{\mathrm{L}} x_{i-1} + \lambda_i^{\mathrm{R}} x_i)$$

$$y = \sum_{i=1}^{n} (\lambda_i^{\mathrm{L}} y_{i-1} + \lambda_i^{\mathrm{R}} y_i)$$

$$\lambda_i^{\mathrm{L}}, \lambda_i^{\mathrm{R}} \geq 0,\ i = 1, \ldots, n$$

$$z_i = \lambda_i^{\mathrm{L}} + \lambda_i^{\mathrm{R}},\ i = 1, \ldots, n$$

$$1 = \sum_{i=1}^{n} z_i$$

$$z_i \in \{0, 1\},\ i = 1, \ldots, n$$

Logarithmic Disaggregated Convex Combination (DLog)

Consists of a logarithmic encoding of the binary variables y_i, which correspond to intervals.

i	$\delta_3\delta_2\delta_1$
1	000
2	001
3	010
4	011
5	100
6	101
7	110
8	111

Let:

- $B_j^0 = \{i : \text{code of } i \text{ has value 0 at position } j\}$.
- $B_j^1 = \{i : \text{code of } i \text{ has value 1 at position } j\}$.

Example:

- $B_1^0 = \{1, 3, 5, 7\}$.
- $B_2^1 = \{3, 4, 7, 8\}$.

Logarithmic Disaggregated Convex Combination (DLog)

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$$\lambda_i^{\mathrm{L}}, \lambda_i^{\mathrm{R}} \geq 0,\ i = 1, \ldots, n$$

$$\sum_{i=1}^{n} (\lambda_i^{\mathrm{L}} + \lambda_i^{\mathrm{R}}) = 1,$$

$$\lambda_i^{\mathrm{L}} + \lambda_i^{\mathrm{R}} \leq \delta_j,\ \ i \in B_j^1,\ j = 1, \ldots, \lceil \log_2 n \rceil,$$

$$\lambda_i^{\mathrm{L}} + \lambda_i^{\mathrm{R}} \leq 1 - \delta_j,\ \ i \in B_j^0,\ j = 1, \ldots, \lceil \log_2 n \rceil,$$

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DCC e DLog

Remarks:

- DLog has the same number of continuous variables and constraints of DCC.
- However DLog needs a logarithmic number of binary variables.

SOS2

A set of variables, let us say $\{\lambda_0, \ldots, \lambda_n\}$, is SOS2 (Special Ordered Set of Variables Type 2) if:

1. At most two variables are positive.
2. If two variables are positive, then they are consecutive in the ordered set, let us say λ_i and λ_{i+1}.

Piecewise-Linear Model Based on SOS2

Givens: $\{(x_0, y_0),\ (x_1, y_1), \ldots, (x_n, y_n)\}$.

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$$1 = \sum_{i=0}^{n} \lambda_i \qquad \lambda_i \geq 0,\ i = 0, \ldots, n$$

$$\{\lambda_i\}_{i=0}^{n} \text{ is SOS2}$$

How Does SOS2 Work?

Implemented directly by the optimization solver:

- Suppose that $\{\lambda_0, \ldots, \lambda_n\}$ is a SOS2 set.
- Let $\{\tilde{\lambda}_0, \ldots, \tilde{\lambda}_n\}$ be the incumbent solution, in which $\tilde{\lambda}_{k_1}, \tilde{\lambda}_{k_2} > 0$ for $k_1, k_2 \in \{0, \ldots, n\}$, $k_1 < k_2$, and $k_2 - k_1 \geq 2$.

The infeasibility can be ruled out by "branching":

- Constraint $\lambda_0 = \cdots = \lambda_{k_1} = 0$ on the left branch.
- Constraint $\lambda_{k_1+2} = \cdots = \lambda_n = 0$ on the right branch.

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Logarithmic Convex Combination (Log)

Remarks:

- Version of the CC model with a logarithmic number of variables and constraints.
- Needs an encoding corresponding to a Gray-Code.
- Complex structure of constraints, particularly in multidimensional domains.
- Requires a domain partitioning given by a J-1 triangulation.

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Log Model: Example

Let $f : \mathbb{R} \to \mathbb{R}$ be a function with a piecewise-linear model:

- Set of breakpoints: $\mathcal{X} = \{x_0, x_1, \ldots, x_n\}$.
- Set of function values: $\mathcal{Y} = \{y_j = f(x_j) : x_j \in \mathcal{X}\}$.

Implementation:

- Set of domain intervals: $\mathcal{I} = \{i_1, \ldots, i_n\}$, such that
  - $i_1 = [x_0, x_1]$, $i_2 = [x_1, x_2]$, ..., $i_n = [x_{n-1}, x_n]$.
- $B : \mathcal{I} \to \{0, 1\}^{\log_2 |\mathcal{I}|}$ is a bijection defining a "Gray Code:"
  - $B(i)$ differs from $B(i+1)$ by just one bit.

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Log Model: Example

$$
\begin{array}{lcl}
B(i_1) = B([x_0, x_1]) & = & (0,0,0) \\
B(i_2) = B([x_1, x_2]) & = & (0,0,1) \\
B(i_3) = B([x_2, x_3]) & = & (0,1,1) \\
B(i_4) = B([x_3, x_4]) & = & (0,1,0) \\
B(i_5) = B([x_4, x_5]) & = & (1,1,0) \\
B(i_6) = B([x_5, x_6]) & = & (1,0,0) \\
B(i_7) = B([x_6, x_7]) & = & (1,0,1) \\
B(i_8) = B([x_7, x_8]) & = & (1,1,1)
\end{array}
$$

Log Model: Example

Let $J^+(B, l) \subseteq \mathcal{X}$ be the subset of breakpoints such that:

- for each $x \in J^+(B, l)$, the interval $I(x) \in \mathcal{I}$ to which it belongs, has value 1 at position l of the binary code $B(I(x))$.

Let $J^0(B, l) \subseteq \mathcal{X}$ be the subset of breakpoints such that:

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Log Model: Example

For $l = 1$:

$$J^+(B,1) = \{x_2, x_7, x_8\}$$
$$J^0(B,1) = \{x_0, x_4, x_5\}$$

For $l = 3$:

$$J^+(B,3) = \{x_5, x_6, x_7, x_8\}$$
$$J^0(B,3) = \{x_0, x_1, x_2, x_3\}$$

For $l = 2$:

$$J^+(B,2) = \{x_3, x_4, x_8\}$$
$$J^0(B,2) = \{x_0, x_1, x_6\}$$

Remark: This structure leads to a "branching scheme" compatible with SOS2.

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For $l = 3$:

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CC Model

$$f = \sum_{v \in \mathcal{V}} f(v)\lambda_v, \qquad x = \sum_{v \in \mathcal{V}} v\lambda_v, \tag{1}$$

$$1 = \sum_{v \in \mathcal{V}} \lambda_v, \tag{2}$$

$$\lambda_v \leq \sum_{P \in \mathcal{P}(v)} y_P, \qquad 1 = \sum_{P \in \mathcal{P}} y_P, \tag{3}$$

$$\lambda_v \geq 0, \; v \in \mathcal{V}, \tag{4}$$

$$y_P \in \{0, 1\}, \; P \in \mathcal{P} \tag{5}$$

Remark: The Log models offers a logarithmic representation of the equations (3) e (5).

Log Model

$$f = \sum_{v \in \mathcal{V}} f(v)\lambda_v, \tag{6}$$

$$x = \sum_{v \in \mathcal{V}} v\lambda_v, \tag{7}$$

$$1 = \sum_{v \in \mathcal{V}} \lambda_v, \tag{8}$$

$$\lambda_v \geq 0,\ v \in \mathcal{V}, \tag{9}$$

$$\sum_{v \in J^+(B,l)} \lambda_v \leq y_l,\ l \in \{1, \ldots, \lceil \log_2 |\mathcal{I}| \rceil\}, \tag{10}$$

$$\sum_{v \in J^0(B,l)} \lambda_v \leq (1 - y_l),\ l \in \{1, \ldots, \lceil \log_2 |\mathcal{I}| \rceil\}, \tag{11}$$

$$y_l \in \{0, 1\},\ l \in \{1, \ldots, \lceil \log_2 |\mathcal{I}| \rceil\}. \tag{12}$$

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- End!
- Thank you for your attention.