Integer Programming: Cutting Planes

Eduardo Camponogara

Department of Automation and Systems Engineering
Federal University of Santa Catarina

October 2016

Summary

Introduction

Agenda

- Study of cutting-plane algorithms that add valid inequalities to the linear relaxation until an integer solution is obtained.
- Gomory cuts, which can be applied to any integer linear program (or mixed-integer).
- Cuts that are specialized for specific problems.

Introduction to Cutting Planes

Integer Problem

The integer problem in general form:

$$IP: \ \max\{c^T x : x \in X\}, \quad \text{where } X = \{x : Ax \leqslant b, x \in \mathbb{Z}_+^n\}$$

Proposition

$conv(X) = \{x : \tilde{A}x \leqslant \tilde{b}, x \geqslant 0\}$ is a polyhedron.

Introduction to Cutting Planes

Integer Problem

The integer problem in general form:

$$IP: \ \max\{c^T x : x \in X\}, \quad \text{where } X = \{x : Ax \leqslant b, x \in \mathbb{Z}^n_+\}$$

Proposition

$conv(X) = \{x : \tilde{A}x \leqslant \tilde{b}, x \geqslant 0\}$ is a polyhedron.

Introduction to Cutting Planes

- The result above states that IP can be reformulated as a linear programming problem:

$$LP: \qquad \max\{c^T x : \tilde{A}x \leqslant \tilde{b}, x \geqslant 0\}$$

- Notice that any extreme point of this LP is an optimal solution of IP.
- For some problems, such as the network flow problem, a complete description of $conv(X)$ is known.

Introduction to Cutting Planes

- In general, and particularly for NP-Hard problems, there is no hope of finding a complete description of $conv(X)$.
- In other situations, such a description can contain an exponential number of constraints/inequalities.
- Given an NP-Hard problem, here the concern is on finding an approximation for $conv(X)$.
- An approximation will be constructed gradually, by adding valid and nontrivial inequalities, preferably inequalities that touch the polyhedron that describes $conv(X)$.

Introduction to Cutting Planes

- In general, and particularly for NP-Hard problems, there is no hope of finding a complete description of $conv(X)$.
- In other situations, such a description can contain an exponential number of constraints/inequalities.
- Given an NP-Hard problem, here the concern is on finding an approximation for $conv(X)$.
- An approximation will be constructed gradually, by adding valid and nontrivial inequalities, preferably inequalities that touch the polyhedron that describes $conv(X)$.

Introduction to Cutting Planes

Valid Inequalities

An inequality $\pi^T x \leqslant \pi_0$ is valid for $X \subseteq \mathbb{R}^n$ if $\pi^T x \leqslant \pi_0$ for all $x \in X$.

Issues

a) Which inequalities are "useful?"

b) If we know a family of valid inequalities for a given problem, how can we use them effectively?

Introduction to Cutting Planes

Valid Inequalities

An inequality $\pi^T x \leqslant \pi_0$ is valid for $X \subseteq \mathbb{R}^n$ if $\pi^T x \leqslant \pi_0$ for all $x \in X$.

Issues

a) Which inequalities are "useful?"
b) If we know a family of valid inequalities for a given problem, how can we use them effectively?

Summary

Introduction to Cutting Planes

Topics

Examples of valid inequalities expressing logic conditions will be presented.

Pure 0-1 Set

- The feasible set X for a 0-1 knapsack problem is given by:

$$X = \{x \in B^5 : 3x_1 - 4x_2 + 2x_3 - 3x_4 + x_5 \leqslant -2\}$$

- For $x_2 = x_4 = 0$, we have the inequality:

$$3x_1 + 2x_3 + x_5 \leqslant -2$$

which becomes impossible to meet.

- Thus, we conclude that a solution must satisfy:

$$x_2 + x_4 \geqslant 1$$

Pure 0-1 Set

- The feasible set X for a 0-1 knapsack problem is given by:

$$X = \{x \in B^5 : 3x_1 - 4x_2 + 2x_3 - 3x_4 + x_5 \leqslant -2\}$$

- For $x_2 = x_4 = 0$, we have the inequality:

$$3x_1 + 2x_3 + x_5 \leqslant -2$$

which becomes impossible to meet.

- Thus, we conclude that a solution must satisfy:

$$x_2 + x_4 \geqslant 1$$

Pure 0-1 Set

- The feasible set X for a 0-1 knapsack problem is given by:

$$X = \{x \in B^5 : 3x_1 - 4x_2 + 2x_3 - 3x_4 + x_5 \leqslant -2\}$$

- For $x_2 = x_4 = 0$, we have the inequality:

$$3x_1 + 2x_3 + x_5 \leqslant -2$$

which becomes impossible to meet.

- Thus, we conclude that a solution must satisfy:

$$x_2 + x_4 \geqslant 1$$

Pure 0-1 Set

- If $x_1 = 1$ and $x_2 = 0$, the following inequality results:

$$2x_3 - 3x_4 + x_5 \leqslant -5$$

which cannot be satisfied.

- Thus:

$$x_1 \leqslant x_2$$

is a valid inequality, which can be introduced in the formulation of X.

Pure 0-1 Set

- If $x_1 = 1$ and $x_2 = 0$, the following inequality results:

$$2x_3 - 3x_4 + x_5 \leqslant -5$$

which cannot be satisfied.

- Thus:

$$x_1 \leqslant x_2$$

is a valid inequality, which can be introduced in the formulation of X.

Pure 0-1 Set

From the above derivations, we can propose a revised formulation for the problem at hand:

$$X = \{x \in B^5 : \begin{array}{l} 3x_1 - 4x_2 + 2x_3 - 3x_4 + x_5 \leqslant -2 \\ x_2 + x_4 \geqslant 1 \\ x_1 \leqslant x_2 \} \end{array}$$

Mixed-Integer 0-1 Set

- An example of mixed-integer (continuous and discrete) set of solutions X is:

$$X = \{(x, y) : x \leqslant 9999y, 0 \leqslant x \leqslant 5, y \in \mathbb{B}\}$$

- It is easy to verify the validity of the inequality $x \leqslant 5y$.

Mixed-Integer 0-1 Set

- An example of mixed-integer (continuous and discrete) set of solutions X is:

$$X = \{(x, y) : x \leqslant 9999y, 0 \leqslant x \leqslant 5, y \in \mathbb{B}\}$$

- It is easy to verify the validity of the inequality $x \leqslant 5y$.

Mixed-Integer 0-1 Set

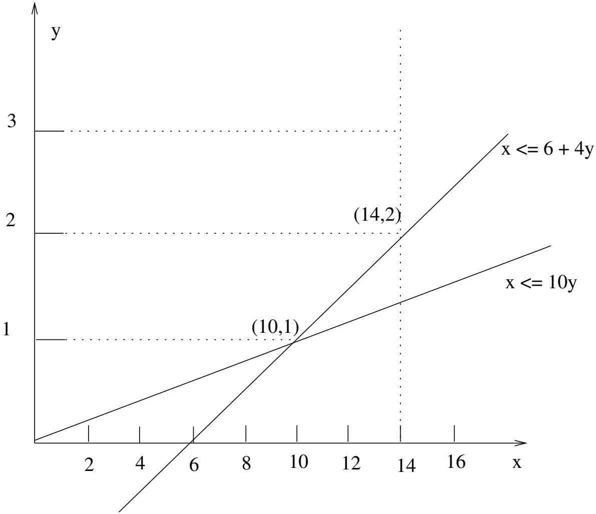
- Consider the set:

$$X = \{(x, y) : 0 \leqslant x \leqslant 10y, 0 \leqslant x \leqslant 14, y \in \mathbb{Z}_+\}$$

- We can verify the validity of the inequality:

$$x \leqslant 14 - 4(2 - y)$$

Mixed-Integer 0-1 Set



Combinatorial Set

Let X be the set of incidence vectors for the matching problem:

$$X = \{x \in \mathbb{Z}_+^{|E|} : \sum_{e \in \delta(i)} x_e \leqslant 1 \qquad \text{for all } i \in V\}$$

where:

- $G = (V, E)$ is an undirected graph;
- $\delta(i) = \{e \in E : e = (i, j) \text{ for some } j \in V\}$.

Combinatorial Set

- Let $T \subseteq V$ be any edge set of odd cardinality.
- The number of edges having both ends in T is at most $(|T| - 1)/2$, therefore we obtain the inequality:

$$\sum_{e \in E(T)} x_e \leqslant \frac{|T| - 1}{2}$$

Combinatorial Set

- $conv(X)$ can be obtained by adding all inequalities of the family above.
- That is, $conv(X)$ is precisely the polyhedron given by:

$$\{x \in \mathbb{R}_+^{|E|} : \quad \sum_{e \in \delta(i)} x_e \leqslant 1 \qquad \forall i \in V$$
$$\sum_{e \in E(T)} x_e \leqslant \frac{|T|-1}{2} \quad \forall T \subseteq V, |T| \text{ odd and } |T| \geqslant 3\}$$

Combinatorial Set

- $conv(X)$ can be obtained by adding all inequalities of the family above.
- That is, $conv(X)$ is precisely the polyhedron given by:

$$\{x \in \mathbb{R}_+^{|E|} : \begin{array}{ll} \sum_{e \in \delta(i)} x_e \leqslant 1 & \forall i \in V \\ \sum_{e \in E(T)} x_e \leqslant \frac{|T|-1}{2} & \forall T \subseteq V, |T| \text{ odd and } |T| \geqslant 3\} \end{array}$$

Integer Rounding

- Consider the regions:

$$\begin{aligned} X &= P \cap \mathbb{Z}^4 \text{ and} \\ P &= \{x \in \mathbb{R}^4_+ : 13x_1 + 20x_2 + 11x_3 + 6x_4 \geqslant 72\} \end{aligned}$$

- Dividing the inequality by 11, we obtain the following valid inequality for P:

$$\frac{13}{11}x_1 + \frac{20}{11}x_2 + \frac{11}{11}x_3 + \frac{6}{11}x_4 \geqslant \frac{72}{11}$$

Integer Rounding

- Consider the regions:

$$\begin{aligned} X &= P \cap \mathbb{Z}^4 \text{ and} \\ P &= \{x \in \mathbb{R}^4_+ : 13x_1 + 20x_2 + 11x_3 + 6x_4 \geqslant 72\} \end{aligned}$$

- Dividing the inequality by 11, we obtain the following valid inequality for P:

$$\frac{13}{11}x_1 + \frac{20}{11}x_2 + \frac{11}{11}x_3 + \frac{6}{11}x_4 \geqslant \frac{72}{11}$$

Integer Rounding

- Since $x \geqslant 0$, we can round the coefficients of x to the nearest integer:

$$
\begin{array}{lllllllll}
\lceil \frac{13}{11} \rceil x_1 & + & \lceil \frac{20}{11} \rceil x_2 & + & x_3 & + & \lceil \frac{6}{11} \rceil x_4 & \geqslant & \frac{72}{11} & \Longrightarrow \\
2x_1 & + & 2x_2 & + & x_3 & + & x_4 & \geqslant & \frac{72}{11} & \Longrightarrow \\
2x_1 & + & 2x_2 & + & x_3 & + & x_4 & \geqslant & \lceil \frac{72}{11} \rceil & \Longrightarrow \\
2x_1 & + & 2x_2 & + & x_3 & + & x_4 & \geqslant & 7 &
\end{array}
$$

- Notice that an integer greater or equal to $6 + \frac{6}{11}$ must be greater or equal to 7.

Integer Rounding

- Since $x \geqslant 0$, we can round the coefficients of x to the nearest integer:

$$
\begin{array}{lclclclcll}
\lceil \frac{13}{11} \rceil x_1 & + & \lceil \frac{20}{11} \rceil x_2 & + & x_3 & + & \lceil \frac{6}{11} \rceil x_4 & \geqslant & \frac{72}{11} & \Longrightarrow \\
2x_1 & + & 2x_2 & + & x_3 & + & x_4 & \geqslant & \frac{72}{11} & \Longrightarrow \\
2x_1 & + & 2x_2 & + & x_3 & + & x_4 & \geqslant & \lceil \frac{72}{11} \rceil & \Longrightarrow \\
2x_1 & + & 2x_2 & + & x_3 & + & x_4 & \geqslant & 7 &
\end{array}
$$

- Notice that an integer greater or equal to $6 + \frac{6}{11}$ must be greater or equal to 7.

Mixed-Integer Rounding

- Consider the example above with the addition of a continuous variable.
- Let $X = P \cap (\mathbb{Z}^4 \times \mathbb{R})$ where:

$$P = \{(y, s) \in \mathbb{R}^4_+ \times \mathbb{R}_+ : 13y_1 + 20y_2 + 11y_3 + 6y_4 + s \geqslant 72\}$$

- Dividing the inequality by 11, we obtain

$$\begin{aligned} \frac{13}{11}y_1 + \frac{20}{11}y_2 + \frac{11}{11}y_3 + \frac{6}{11}y_4 + \frac{s}{11} &\geqslant \frac{72}{11} \Longrightarrow \\ \frac{13}{11}y_1 + \frac{20}{11}y_2 + \frac{11}{11}y_3 + \frac{6}{11}y_4 &\geqslant \frac{72-s}{11} \end{aligned}$$

Mixed-Integer Rounding

- Consider the example above with the addition of a continuous variable.
- Let $X = P \cap (\mathbb{Z}^4 \times \mathbb{R})$ where:

$$P = \{(y, s) \in \mathbb{R}^4_+ \times \mathbb{R}_+ : 13y_1 + 20y_2 + 11y_3 + 6y_4 + s \geqslant 72\}$$

- Dividing the inequality by 11, we obtain

$$\frac{13}{11}y_1 + \frac{20}{11}y_2 + \frac{11}{11}y_3 + \frac{6}{11}y_4 + \frac{s}{11} \geqslant \frac{72}{11} \Longrightarrow$$
$$\frac{13}{11}y_1 + \frac{20}{11}y_2 + \frac{11}{11}y_3 + \frac{6}{11}y_4 \geqslant \frac{72-s}{11}$$

Mixed-Integer Rounding

- We can observe that:

$$\begin{array}{lllllllllllll} 2y_1 & + & 2y_2 & + & y_3 & + & y_4 & \geqslant & \lceil \frac{72}{11} \rceil & = & 7 & \text{se } s = 0 \\ 2y_1 & + & 2y_2 & + & y_3 & + & y_4 & \geqslant & \lceil \frac{72-6}{11} \rceil & = & 6 & \text{se } s = 6 \end{array}$$

- This suggests the following valid inequality:

$$2y_1 + 2y_2 + y_3 + y_4 + \alpha s \geqslant 7$$

for some α.

The above inequality is valid for $\alpha \geqslant \frac{1}{6}$.

Summary

Theory of Valid Inequalities

The concepts on valid inequalities will be investigated in more depth.

Valid Inequalities for Linear Programs

- Consider the polyhedron:

$$P = \{x : Ax \leqslant b, x \geqslant 0\}$$

and the inequality:

$$\pi^T x \leqslant \pi_0.$$

- Is the inequality (π, π_0) valid for P?

Valid Inequalities for Linear Programs

- Consider the polyhedron:

$$P = \{x : Ax \leqslant b, x \geqslant 0\}$$

and the inequality:

$$\pi^T x \leqslant \pi_0.$$

- Is the inequality (π, π_0) valid for P?

Valid Inequalities for Linear Programs

Proposition

$\pi^T x \leqslant \pi_0$ is valid for $P = \{x : Ax \leqslant b, x \geqslant 0\} \neq \emptyset$ if, and only if,

a) there exists $u \geqslant 0$ and $v \geqslant 0$ such that $u^T A - v^T = \pi^T$ and $u^T b \leqslant \pi_0$, or

b) there exists $u \geqslant 0$ such that $u^T A \geqslant \pi^T$ and $u^T b \leqslant \pi_0$.

Valid Inequalities for Linear Programs

Proof (b)

If there exists $u \geqslant 0$ such that $u^T A \geqslant \pi^T$ and $u^T b \leqslant \pi_0$, then any $x \in P$,

$$\begin{aligned} Ax \leqslant b &\Longrightarrow u^T Ax \leqslant u^T b \\ &\Longrightarrow \pi^T x \leqslant u^T Ax \leqslant u^T b \leqslant \pi_0 \\ &\Longrightarrow (\pi, \pi_0) \text{ is a valid inequality.} \end{aligned}$$

■

Valid Inequalities for Integer Programs

Proposition

The inequality $y \leqslant \lfloor b \rfloor$ is valid for $X = \{y \in \mathbb{Z} : y \leqslant b\}$.

Valid Inequalities for Integer Programs

Example

We can use the proposition above to generate valid inequalities for the polyhedron given by the following inequalities:

$$\begin{aligned} 7x_1 - 2x_2 &\leqslant 14 \\ x_2 &\leqslant 3 \\ 2x_1 - 2x_2 &\leqslant 3 \\ x &\geqslant 0, \quad x \text{ integer} \end{aligned}$$

Valid Inequalities for Integer Programs

Example

i) Multiplying the constraint by a vector of nonnegative values $u = (\frac{2}{7}, \frac{37}{63}, 0)$, we obtain a valid inequality:

$$2x_1 + \frac{1}{63}x_2 \leqslant \frac{121}{21}$$

ii) Reducing the coefficients on the left-hand side to the nearest integer, we obtain:

$$2x_1 + 0x_2 \leqslant \frac{121}{21}$$

Valid Inequalities for Integer Programs

Example

i) Multiplying the constraint by a vector of nonnegative values $u = (\frac{2}{7}, \frac{37}{63}, 0)$, we obtain a valid inequality:

$$2x_1 + \frac{1}{63}x_2 \leqslant \frac{121}{21}$$

ii) Reducing the coefficients on the left-hand side to the nearest integer, we obtain:

$$2x_1 + 0x_2 \leqslant \frac{121}{21}$$

Valid Inequalities for Integer Programs

Example

iii) Since the left-hand side assumes integer values, we can reduce the right-hand side to the nearest integer, leading to another inequality:

$$2x_1 \leqslant \lfloor \frac{121}{21} \rfloor = 5 \implies x_1 \leqslant \frac{5}{2} \implies x_1 \leqslant 2$$

Chvátal-Gomory Procedure

- The CG (Chvátal-Gomory) procedure formalizes the steps followed above, to generate *all valid inequalities of an integer program*.
- Let $X = P \cap \mathbb{Z}^n$ be a set of solutions where:
  - $P = \{x \in \mathbb{R}^n_+ : Ax \leqslant b\}$ is a polyhedron, and
  - $A \in \mathbb{R}^{m \times n}$ is a matrix with columns $\{a_1, a_2, \ldots, a_n\}$.

Chvátal-Gomory Procedure

- The CG (Chvátal-Gomory) procedure formalizes the steps followed above, to generate *all valid inequalities of an integer program*.
- Let $X = P \cap \mathbb{Z}^n$ be a set of solutions where:
  - $P = \{x \in \mathbb{R}^n_+ : Ax \leqslant b\}$ is a polyhedron, and
  - $A \in \mathbb{R}^{m \times n}$ is a matrix with columns $\{a_1, a_2, \ldots, a_n\}$.

Chvátal-Gomory Procedure

Given $u \in \mathbb{R}^m_+$, the procedure consists of the following steps:

Step 1: the inequality:

$$\sum_{j=1}^{n} u^T a_j x_j \leqslant u^T b$$

is valid for P because $u \geqslant 0$ and $\sum_{j=1}^{n} a_j x_j \leqslant b$.

Chvátal-Gomory Procedure

Step 2: The inequality:

$$\sum_{j=1}^{n} \lfloor u^T a_j \rfloor x_j \leqslant u^T b$$

is valid for P since $x \geqslant 0$.

Chvátal-Gomory Procedure

Step 3: The inequality

$$\sum_{j=1}^{n} \lfloor u^T a_j \rfloor x_j \leqslant \lfloor u^T b \rfloor$$

is valid for P since x is integer and further because

$$\sum_{j=1}^{n} \lfloor u^T a_j \rfloor x_j$$

is integer.

Chvátal-Gomory Procedure

Important

The fact that the CG procedure can yield all valid inequalities of an integer program is of major relevance.

Chvátal-Gomory Procedure

Theorem

Every valid inequality for X can be obtained by applying the Chvátal-Gomory procedure for a finite number of times.

Cutting Planes

- Thank you for attending this lecture!!!