Fundamentals

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Summary

Optimization

Definition

Discipline concerned with:

- the computation of values for decision variables
  that induce optimal performance and
  which satisfy constraints of a mathematical model.

Problem Modeling

Need of representing the real-world:

- impossibility to interact directly with the world,
- economic hurdles,
- complexity.

We seek a representation of the world by means of well-structured models that are representative of the reality.

Problem Modeling

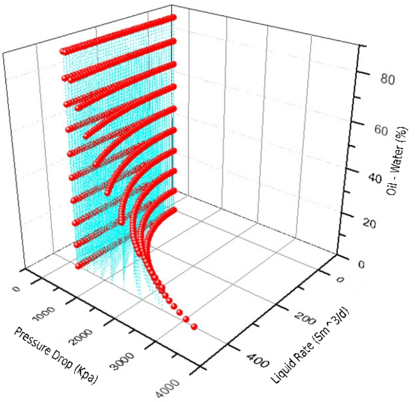
Models are simplified representations of the reality that, under some conditions and situations, are sufficiently representative.

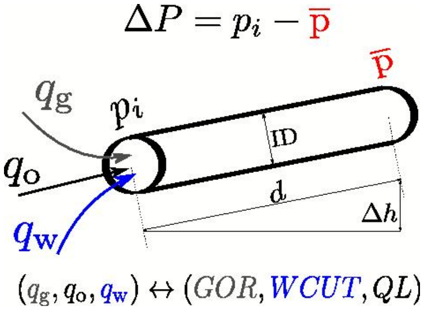
Model Characteristics

Desired characteristics:

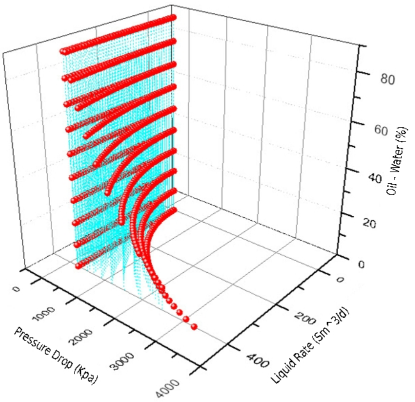
- representation of the world;
- simplicity.

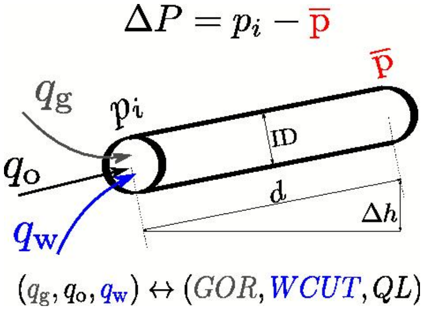
Modeling: Pressure Drop in Flowlines





Modeling: Pressure Drop in Flowlines





Optimization Models

Models:

- Aim to maximize a performance criterion, such oil and gas production,
- subject to constraints that define the operating envelope.

Optimization problems are define in "Mathematical Programming," declarative language which is universally adopted.

Elements of an Optimization Formulation

1) Decision variables
2) Objective function
3) Constraints

General Formulation

$$
\begin{array}{ll}
\text{Minimize} & f(x) \\
\text{Subject to :} & \\
 & g(x) \leqslant 0 \\
 & h(x) = 0 \\
 & x \in \mathbb{R}^n
\end{array}
$$

Summary

Linear Combination

- Let $X = \{x_1, \ldots, x_n\}$ be a set of vectors in $\mathbb{R}^n$;
- A vector x is said to be a linear combination of X if:

$$x = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$$
$$\alpha_1, \ldots, \alpha_n \in \mathbb{R}$$

Affine Combination

- Let $X = \{x_1, \dots, x_n\}$ be a set of vectors in $\mathbb{R}^n$;
- A vector x is said to be an affine combination of X if:

$$x = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$$
$$\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$$
$$\alpha_1, \dots, \alpha_n \in \mathbb{R}$$

Convex Combination

- Let $X = \{x_1, \ldots, x_n\}$ be a set of vectors in $\mathbb{R}^n$;
- A vector x is said to be a convex combination of X if:

$$
\begin{gathered}
x = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n \\
\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1 \\
\alpha_1, \ldots, \alpha_n \geq 0
\end{gathered}
$$

Vector Space

A set of vectors S is a vector space if:

- closed with respect to addition:

$$x + y \in S, \ \forall x, y \in S$$

- closed with respect to scalar multiplication:

$$\alpha x \in S, \ \forall x \in S, \ \forall \alpha \in \mathbb{R}$$

Affine Hull

Affine Hull of a set of vectors $X \subset \mathbb{R}^n$ is the set

$$\mathrm{aff}(X) = \{x : x \text{ is an affine combination of elements of } X\}$$

- Example: $\{x \in \mathbb{R}^n : Ax = b\}$ is an affine set.
- Exercise: prove this property.

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Linear Function

A function $f : \mathbb{R}^n \to \mathbb{R}$ is linear if:

- $f(\alpha x) = \alpha f(x)$ for all $x \in \mathbb{R}^n, \ \alpha \in \mathbb{R}$.
- $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}^n$.

Taylor's Expansion

- Let $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function.
- Then:

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)(\widehat{x})}(x - \widehat{x})^k}{k!} + \mathcal{O}(|x - \widehat{x}|^{n+1})$$

- A polynomial is a universal approximator.

Taylor's Expansion

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function. Then:

$$f(x) = f(\widehat{x}) + \nabla f(\widehat{x})^{\mathrm{T}}(x - \widehat{x}) + \frac{1}{2}(x - \widehat{x})^{\mathrm{T}}\nabla^2 f(\widehat{x})(x - \widehat{x}) + \mathcal{O}(\|x - \widehat{x}\|^3)$$

in which:

- $\nabla f(\widehat{x})$ is the gradient; and
- $\nabla^2 f(\widehat{x})$ is the Hessian matrix.

Convex Function

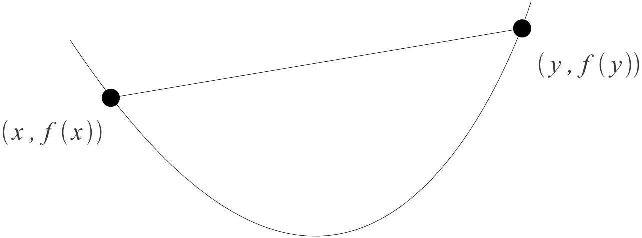
A function $f : R^n \rightarrow R$ is convex if:

- if the domain of f is a convex set and
- for all $x, y \in \operatorname{dom} f$, and θ such that $0 \leq \theta \leq 1$, we have that:

$$f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y)$$

Convex Function

Geometrically, the inequality means that the chord between $(x, f(x))$ and $(y, f(y))$ lies above the graph of f:



Convex Function: Example

Consider a system of linear equations:

$$Ax = b$$

If $b \notin \mathrm{range}(A) \iff \mathrm{rank}(A) < \mathrm{rank}([A\ b])$, then an approximate solution can be sought by solving a least squares problem:

$$\min_x f(x) = \|Ax - b\|^2$$

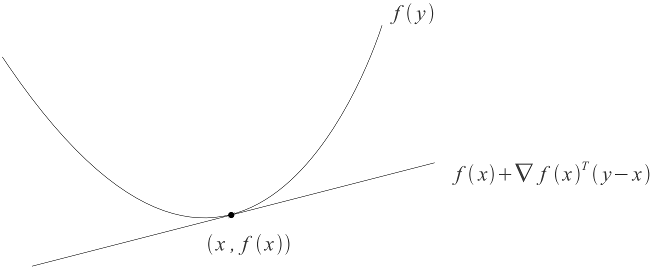
Notice that $f(x)$ is a convex function.

Convex Function: First-Order Condition

If f is differential, then the f is convex if and only if $\mathrm{dom}\, f$ is a convex set and

$$f(y) \geq f(x) + \nabla f(x)^{\mathrm{T}}(y - x)$$

for all $x, y \in \mathrm{dom}\, f$.



Convex Function: Second-Order Condition

If f is differential, then the f is convex if and only if $\mathrm{dom}\, f$ is a convex set and its Hessian is positive semidefinite:

$$\nabla^2 f(x) \succeq 0 \quad \forall x \in \mathrm{dom}\, f$$

Exercise: show that the least squares problem is convex.

Summary

Introduction

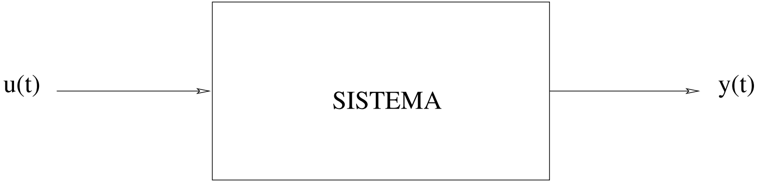
Fundamentals

System Identification

System Identification

- Propose a model of a system whose internal elements and connections are unknown.
- Only input signals and corresponding outputs are given.
- Carry out experiments by introducing input values $u(t), t = 0, \ldots, N$, and then observing the resulting outputs $y(t), t = 0, \ldots, N$ over time.

System Identification



System Identification

A simple and widely used model is the moving average:

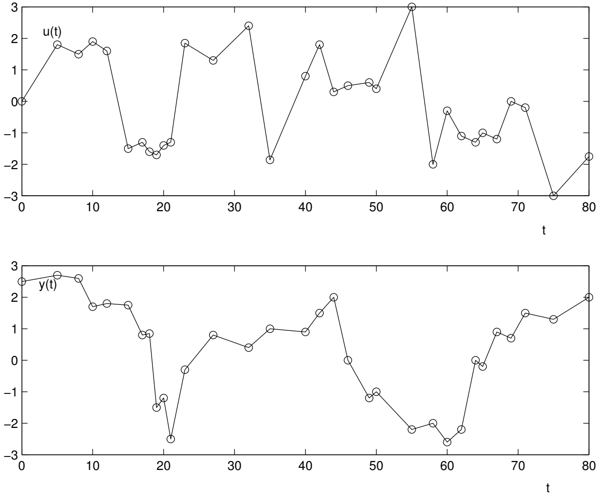
$$\hat{y}(t) = h_0 u(t) + h_1 u(t-1) + h_2 u(t-2) + \cdots + h_n u(t-n)$$

with n delays.

System Identification

- $\hat{y}(t)$ is the prediction of $y(t)$ produced by the model for the current input, $u(t)$,, and past n inputs, $u(t-1), u(t-2), \ldots, u(t-n)$.
- $h_0, h_1, \ldots, h_n$ are the parameters that define the linear combination of the inputs.

Input-Output Example



Least Squares Estimation

Find a model that minimizes the prediction error:

$$\min_{h_0, \ldots, h_n} E = \left[\sum_{t=n}^{t=N} (\hat{y}(t) - y(t))^2 \right]^{\frac{1}{2}}$$

Least Squares Estimation

Notice that the error E can be expressed in matrix form, by letting:

$$\begin{aligned} E &= \left[\sum_{t=n}^{t=N} (\hat{y}(t) - y(t))^2\right]^{\frac{1}{2}} \\ &= \|Ax - b\| \end{aligned}$$

$$A = \begin{bmatrix} u(n) & u(n-1) & u(n-2) & \dots & u(0) \\ u(n+1) & u(n) & u(n-1) & \dots & u(1) \\ u(n+2) & u(n+1) & u(n) & \dots & u(2) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ u(N) & u(N-1) & u(N-2) & \dots & u(N-n) \end{bmatrix}$$

Least Squares Estimation

$$x = \begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ \vdots \\ h_n \end{bmatrix} \text{ e } b = \begin{bmatrix} y_n \\ y_{n+1} \\ y_{n+2} \\ \vdots \\ y_N \end{bmatrix}$$

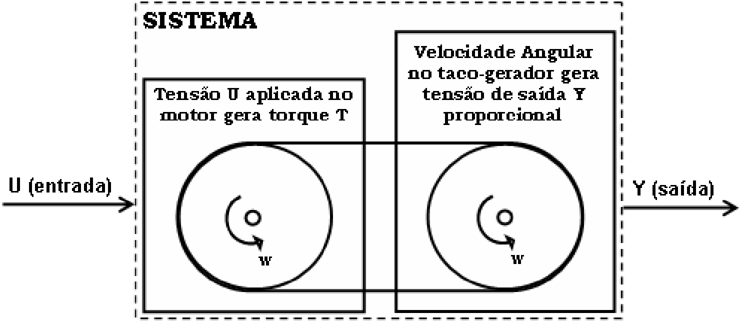
Least Squares Estimation

The estimation problem can be cast as least squares problem:

$$\min_{x \in \mathbb{R}^{n+1}} \|Ax - b\|^2$$

System Identification of a Simple Motor-Generator

- Generator connected to a motor by a belt.



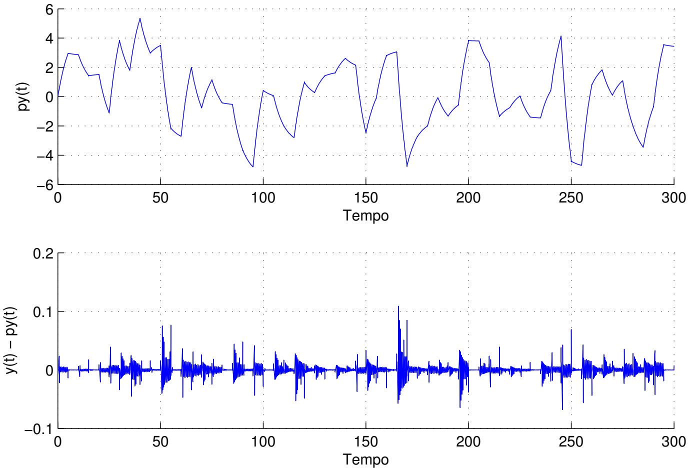
System Identification of a Simple Motor-Generator

Prediction model takes into account past outputs:

$$\begin{aligned}\hat{y}(t) = h_0 u(t) + h_1 u(t-1) + h_2 u(t-2) + \cdots + h_n u(t-n) + \\ + w_1 y(t-1) + w_2 y(t-2) + \cdots + w_n y(t-n)\end{aligned}$$

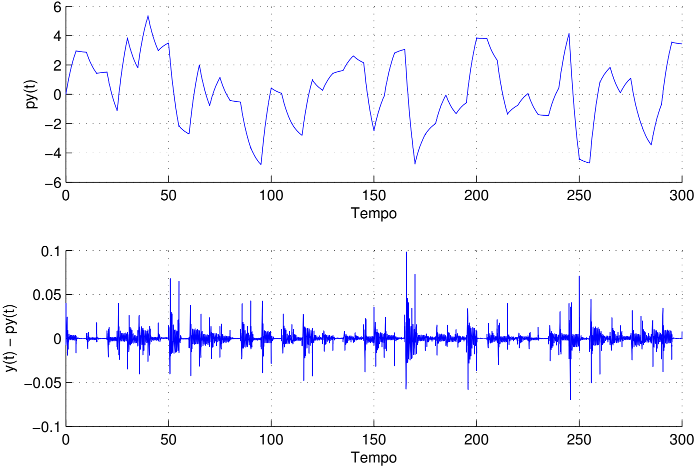
This models uses feedback, whereas a model purely based on inputs would be an open-loop model.

System Identification ($n = 3$)



The errror vector $e = y(t) - \hat{y}(t)$ has norm $\|e\| = 0.5495$.

System Identification ($n = 10$)



The error vector $e = y(t) - \hat{y}(t)$ has norm $\|e\| = 0.4778$.

Fundamentals

- Thank you for attending this lecture!!!