Set Invariance for Delay Difference Equations ⋆

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Abstract: This paper deals with set invariance for time delay systems. The first goal of the paper is to review the known necessary or sufficient conditions for the existence of invariant sets with respect to dynamical systems described by discrete-time delay difference equations (dDDEs). Secondly, we address the construction of invariant sets in the original state space (also called $\mathcal{D}$-invariant sets) by exploiting the forward mappings.
As novelties, the present paper contains a sufficient condition for the existence of ellipsoidal $\mathcal{D}$-contractive sets for dDDEs, and a necessary and sufficient condition for the existence of $\mathcal{D}$-invariant sets in relation to time-varying dDDE stability. Another contribution is the clarification of the relationship between convexity (convex hull operation) and $\mathcal{D}$-*invariance*. In short, it is shown that the convex hull of two $\mathcal{D}$-*invariant* sets is not $\mathcal{D}$-*invariant* but the convex hull of a non-convex $\mathcal{D}$-*invariant* set is $\mathcal{D}$-*invariant*.

Keywords: Set invariance, Time-delay systems, discrete-time Delay Difference Equations.

1. INTRODUCTION

Positive invariance is an essential concept in control theory, with applications in constrained control analysis, uncertainty handling and design problems (Blanchini (1999), Blanchini and Miani (2008)). It serves as basic tool in many topics, such as model predictive control (Mayne et al. (2000)) and fault tolerant control (Olaru et al. (2010)).

The response of a dynamical system to external excitation is rarely instantaneous. The time-delay offers the appropriate modeling framework for such propagation phenomena. Time-delay systems have been considered in different control applications (see for example the recent results by Avila Alonso et al. (2014); Boussaada et al. (2012); Seuret et al. (2014)).

Delay difference equations (DDEs) form an important modeling class, since most modern controllers are implemented via computers or dedicated embedded systems. Two main approaches exist in the literature dealing with positive invariant sets for discrete-time delay difference equations (dDDEs). The first approach is enabled by the fact that the discrete-time DDE allows a finite-dimensional extended state space model. This extended state space, whose dimension is finite but in direct relation with the delay value, leads to invariant set characterization with respect to an equivalent linear time-invariant model. This concept is well understood and popular in the literature, but it suffers from an increased numerical complexity when delays are relatively large. The second approach has been formulated in the '90s and re-investigated in the last decade, to obtain an invariant set for the DDE in the original state space, which is independent of the delay value. This concept is also denoted as $\mathcal{D}$-invariance, and is often conservative as long as the existence conditions are restrictive. The link between the two representations and their invariant sets has received recently a unifying characterization via set factorization - Olaru et al. (2014).

In this paper we address the existence of positive invariant sets in the state space of the original dDDE, which are also referred to as $\mathcal{D}$-invariant sets and which can be seen as invariant sets in both the current and the retarded state space and further related to the stability based on Lyapunov-Razumikhin approach. A necessary and sufficient characterization for the existence of $\mathcal{D}$-invariant sets was provided by Hennet and Tarbouriech (1998); Vassilaki and Bitsoris (1999). Particularly, as regards the construction of $\mathcal{D}$-invariant sets, we can find a series of results by Lombardi et al. (2011b,a). We provide in the present paper an interesting example for which the condition by Stankovic et al. (2014) is verified but the existing algorithms fail to construct a $\mathcal{D}$-invariant set.

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As main contributions we : i) propose a sufficient condition for the existence of ellipsoidal $\mathcal{D}$-invariant sets for dDDEs; ii) establish the relationship between time-varying dDDE stability and the existence of $\mathcal{D}$-invariant sets; iii) prove two properties related to convexity and convex operations over $\mathcal{D}$-invariant sets. Notably, it is established that a dDDE admits a $\mathcal{D}$-invariant set if and only if it is time-varying-delay independent stable.

This paper is structured as follows. Section 2 presents some preliminary mathematical notions and definitions. Basic properties of $\mathcal{D}$-invariance concept are addressed in Section 3. In the same section we present necessary and sufficient conditions for the existence of non trivial sets, the relationship between $\mathcal{D}$-invariance and stability of dDDEs concludes the section. Algorithmic construction based on set iteration using the forward mappings, and some illustrative examples are revisited in Section 4, and finally Section 5 draws some concluding remarks.

2. DEFINITION AND CONSIDERED DYNAMICS

2.1 Notations

We denote by $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$ and $\mathbb{Z}_+$ sets of real numbers, non-negative reals, integer numbers and non-negative integers, respectively. For an arbitrary set $\mathcal{A} \subseteq \mathbb{R}^n$, $int(\mathcal{A})$ denotes the interior of $\mathcal{A}$. $\mathbb{B}^n_r(0)$ denotes the ball of radius r in Euclidean norm, centered in the origin of $\mathbb{R}^n$. $\oplus$ denotes the Minkowski sum of sets.

Definition 1. A set $\mathcal{P} \subseteq \mathbb{R}^n$ *is bounded if* there exists $r \in \mathbb{R}_+$ such that $\mathcal{P} \subset \mathbb{B}^n_r(0)$.

Definition 2. A set $\mathcal{P} \subseteq \mathbb{R}^n$ *is closed if* $\forall x \notin \mathcal{P}$ there exists $\epsilon \in \mathbb{R}_+$ *such that* $\mathbb{B}^n_\epsilon(x) \cap \mathcal{P} = \emptyset$.

Definition 3. A set $\mathcal{P} \subseteq \mathbb{R}^n$ *is compact if it is bounded and closed.*

Definition 4. A set $\mathcal{P} \subseteq \mathbb{R}^n$ *is a (proper)* $\mathcal{C}$-set *if is convex, compact and includes the origin in its strict interior.*

We denote by $Com(\mathbb{R}^n)$ and $ComC(\mathbb{R}^n)$ the space of compact subsets and the space of $\mathcal{C}$-subsets of $\mathbb{R}^n$ containing the origin, respectively. The spectrum of a matrix $A \in \mathbb{R}^{n\times n}$ is the set of the eigenvalues of A, denoted by $\lambda(A)$, while the spectral radius is defined as $\rho(A) := \max_{\xi\in\lambda(A)}(|\xi|)$. The spectral norm will be denoted by $\sigma(A)$ and is defined as $\sigma(A) := \sqrt{\rho(A^TA)}$.

2.2 System Dynamics

In the present paper we will consider discrete-time delay difference equations in the form:

$$x(k+1) = A_0x(k) + A_dx(k-d) \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state vector at the time $k \in \mathbb{Z}_+$, $d \in \mathbb{Z}_+$ is the *fixed* time-delay, the matrices $A_j \in \mathbb{R}^{n\times n}$, for $j \in \mathbb{Z}_{\{0,d\}}$ and the initial conditions are considered to be given by $x(-i) = x_{-i} \in \mathbb{R}^n$, for $i \in \mathbb{Z}_{\{0,d\}}$.

3. $\mathcal{D}$-INVARIANCE PROPERTIES

3.1 Definitions

Definition 5. A set $\mathcal{P} \subseteq \mathbb{R}^n$ is called $\mathcal{D}$-invariant for the system (1) with initial conditions $x_{-i} \in \mathcal{P}$ for all $i \in \mathbb{Z}_{[0,d]}$ if the state trajectory satisfies $x_k \in \mathcal{P}, \forall k \in \mathbb{Z}_+$. □

Lemma 6. The following statements are equivalent:

- i $\mathcal{P} \subseteq \mathbb{R}^n$ is $\mathcal{D}$-invariant for system (1).
- ii $A_0\mathcal{P} \oplus A_d\mathcal{P} \subseteq \mathcal{P}$.

Several properties fix a set of basic relations between $\mathcal{D}$-invariant sets.

Proposition 7. The following properties hold:

- i If $\mathcal{P} \in \mathbb{R}^n$ is $\mathcal{D}$-invariant then $\alpha\mathcal{P}$ is $\mathcal{D}$-invariant for any $\alpha \in \mathbb{R}_{>0}$.
- ii Let $\mathcal{P}_1, \mathcal{P}_2 \subseteq \mathbb{R}^n$ be two $\mathcal{D}$-invariant sets for (1). Then $\mathcal{P}_1 \cap \mathcal{P}_2$ is a $\mathcal{D}$-invariant set for the same dynamical system.
- iii If for some $d_1, d_2 \in \mathbb{N}_+$, $d_1 \neq d_2$, the set $\mathcal{P} \in \mathbb{R}^n$ is $\mathcal{D}$-invariant for the system
$$x(k+1) = A_0x(k-d_1) + A_dx(k-d_2) \tag{2}$$
then $\mathcal{P}$ is $\mathcal{D}$-invariant for
$$x(k+1) = A_0x(k-\bar{d}_1) + A_dx(k-\bar{d}_2) \tag{3}$$
for any $\bar{d}_1, \bar{d}_2 \in \mathbb{N}_+$.
- iv The convex hull of $\mathcal{D}$-invariant sets is not necessarily $\mathcal{D}$-invariant.

Proof. For the proof of Lemma 6 and Properties i, ii and iii of Proposition 7 see Lombardi (2011). For property iv of Proposition 7, consider the system:

$$x(k+1) = \begin{bmatrix} 0.2 & 0.01 \\ 0 & 0.7 \end{bmatrix} x(k) + \begin{bmatrix} 0.6 & 0 \\ 0.005 & 0.25 \end{bmatrix} x(k-1), \tag{4}$$

Then the set

$$D_1 = \left\{ x \in \mathbb{R}^2 | \begin{bmatrix} -0.1 \\ -1 \end{bmatrix} \leq x \leq \begin{bmatrix} 0.1 \\ 1 \end{bmatrix} \right\}$$

is $\mathcal{D}$-invariant as well as

$$D_2 = \left\{ x \in \mathbb{R}^2 | \begin{bmatrix} -1 \\ -0.1 \end{bmatrix} \leq x \leq \begin{bmatrix} 1 \\ 0.1 \end{bmatrix} \right\}$$

On the other hand the set obtained as convex hull $D = Conv(D_1, D_2)$ is not $\mathcal{D}$-invariant. □

3.2 Necessary conditions for $\mathcal{D}$-invariance

Basic algebraic conditions Let us introduce the following notation for the extended state-space matrix:

$$A_\xi = \begin{bmatrix} A_0 & 0 & \dots & 0 & A_d \\ I & 0 & \dots & 0 & 0 \\ 0 & I & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & I & 0 \end{bmatrix} \tag{5}$$

Proposition 8. (Lombardi et al. (2011b)) Considering the system (1), the existence of a $\mathcal{D}$-invariant $\mathcal{C}$-set $\mathcal{P}$ implies that:

- i the spectral radii of the matrices A_0, A_d and A_ξ are subunitary: $\rho(A_i) \leq 1$, $\forall i \in \{0, d, \xi\}$;
- ii the spectral radius of the matrix $(A_0 + A_d)$ is subunitary: $\rho(A_0 + A_d) \leq 1$.

Specific algebraic conditions for 2 delay dDDEs For dDDEs with two delay parameters, Stankovic et al. (2014) recently provided a computationally efficient numerical condition which is necessary to guarantee the existence of Lyapunov-Razumikhin contractive sets. This test is sufficient for the robust asymptotic stability [1] with respect to the delay parameter and can be employed in the $\mathcal{D}$-invariance context. We denote by $\mathbb{D}$, $\partial\mathbb{D}$ the open unit disc and the unit circle, respectively. For the matrix pair (A, B), the set of generalized eigenvalues and the Kronecker product are denoted by $\gamma(A, B)$ and $A \otimes B$, respectively. $I_n \in \mathbb{R}^{n\times n}$ and $0_{n\times m} \in \mathbb{R}^{n\times m}$ denote the identity and the null matrix, respectively. The main result is in the next theorem.

Theorem 9. (Stankovic et al. (2014)) Assume that $\rho(A_0 + A_d) \le 1$. Then, system (1) admits a $\mathcal{D}$-contractive set only if $\gamma(U, V) \cap \partial\mathbb{D} = \emptyset$, where

$$U = \begin{pmatrix} 0_{n^2\times n^2} & I_{n^2} \\ -B_0 & -B_1 \end{pmatrix}, V = \begin{pmatrix} I_{n^2} & 0_{n^2\times n^2} \\ 0_{n^2\times n^2} & B_2 \end{pmatrix} \quad (6)$$

$$B_0 = A_0 \otimes A_d^T, B_1 = A_0 \otimes A_0^T + A_d \otimes A_d^T - I_{n^2}, B_2 = A_d \otimes A_0^T \quad (7)$$

As stated by Stankovic et al. (2014), the condition of Theorem 9 covers the existing necessary conditions for the two delay case. However, we report here an interesting example which points out the possible limitation of this condition.

Example 10. Consider the system (1) with:

$$A_0 = \begin{pmatrix} 0.5 & 0.5 \\ 0 & 0 \end{pmatrix}; \quad A_d = \begin{pmatrix} 0 & 0.5 \\ -0.5 & 0.5 \end{pmatrix} \quad (8)$$

For this numerical example, one can compute:

$$\rho(A_0 + A_d) = 0.8660 < 1$$

and

$$\gamma(U, V) = 1.7442 \pm 1.9433\mathrm{i}, 0.2558 \pm 0.2850\mathrm{i}, 0, 0, inf, inf.$$

The necessary condition proposed in Stankovic et al. (2014) is fulfilled. However, up to the existing constructive routines (see next section) there is no numerical construction able to determine a $\mathcal{D}$-invariant set for this system. □

3.3 Sufficient conditions for $\mathcal{D}$-invariance

The converse problem of establishing sufficient conditions for the existence of $\mathcal{D}$-invariant sets was stated by Lombardi (2011) with two tests that concentrate on the spectral norms of the matrices appearing in the dDDE (1). A different approach for establishing sufficient conditions is to exploit the structural properties of specific classes of candidate $\mathcal{D}$-invariant sets. We propose next a contribution in this sense with a sufficient condition for the existence of ellipsoidal $\mathcal{D}$-contractive sets for a dDDE. As it is often the case in this framework, the test are based on LMIs.

[1] See Stankovic et al. (2014) for a formal definition of *robust asymptotic stability*.

Theorem 11. Considering the dynamical system:

$$x_{k+1} = A_0 x_{k-d_0} + A_1 x_{k-d_1}, \quad (9)$$

the existence of an ellipsoidal $\mathcal{D}$-invariant set is guaranteed if the following two LMIs hold for some $P = P^T \succ 0$:

$$\begin{pmatrix} A_0^T P A_0 - P & A_0^T P A_1 \\ A_1^T P A_0 & A_1^T P A_1 \end{pmatrix} \prec 0 \quad (10)$$

$$\begin{pmatrix} A_0^T P A_0 & A_0^T P A_1 \\ A_1^T P A_0 & A_1^T P A_1 - P \end{pmatrix} \prec 0. \quad (11)$$

Proof. The set

$$\Psi = \{x \in \mathbb{R}^n, x^T P x \le 1\}$$

is $\mathcal{D}$-invariant for the system described by the dDDE (9)if $x_{k+1} \in \Psi$, $\forall x_{k-d_0}, x_{k-d_1} \in \Psi$. This is equivalent to the simultaneous verification of the two inequalities :

$$x_{k+1}^T P x_{k+1} - x_{k-d_0}^T P x_{k-d_0} < 0, \quad (12)$$

$$x_{k+1}^T P x_{k+1} - x_{k-d_1}^T P x_{k-d_1} < 0 \quad (13)$$

Exploiting the dDDE relationship one has:

$$\begin{aligned} x_{k+1}^T P x_{k+1} - x_{k-d_0}^T P x_{k-d_0} &= \\ (A_0 x_{k-d_0} + A_1 x_{k-d_1})^T P (A_0 x_{k-d_0} + A_1 x_{k-d_1}) &= \\ x_{k-d_0}^T (A_0^T P A_0 - P) x_{k-d_0} + x_{k-d_1}^T (A_1^T P A_1) x_{k-d_1} &+ \\ x_{k-d_0}^T (A_0^T P A_1) x_{k-d_1} + x_{k-d_1}^T (A_1^T P A_0) x_{k-d_0} &< 0 \end{aligned}$$

and in the equivalent matrix formulation:

$$\begin{pmatrix} x_{k-d_0}^T & x_{k-d_1}^T \end{pmatrix} \begin{pmatrix} A_0^T P A_0 - P & A_0^T P A_1 \\ A_1^T P A_0 & A_1^T P A_1 \end{pmatrix} \begin{pmatrix} x_{k-d_0} \\ x_{k-d_1} \end{pmatrix} \prec 0 \quad (14)$$

Analogously for the second inequality:

$$\begin{pmatrix} x_{k-d_0}^T & x_{k-d_1}^T \end{pmatrix} \begin{pmatrix} A_0^T P A_0 & A_0^T P A_1 \\ A_1^T P A_0 & A_1^T P A_1 - P \end{pmatrix} \begin{pmatrix} x_{k-d_0} \\ x_{k-d_1} \end{pmatrix} \prec 0 \quad (15)$$

We can conclude that the existence of a positive definite matrix $P = P^T$ is a sufficient condition for the existence of an ellipsoidal $\mathcal{D}$-invariant set, and the proof is complete. □

For illustration let us consider the system (9) with:

$$A_0 = \begin{pmatrix} 0.35 & 0.13 \\ 0.51 & -0.01 \end{pmatrix}, A_1 = \begin{pmatrix} 0.51 & -0.01 \\ 0.03 & 0.51 \end{pmatrix}. \quad (16)$$

The condition for the existence of a $\mathcal{D}$-contractive set proposed in Theorem 11 is fulfilled and the $\mathcal{D}$-contractive set exists as shown in Figure 1. It is interesting to note that the sufficient condition $\|A_0\|_p + \|A_1\|_p \le 1$ by Hennet and Tarbouriech (1998); Lombardi et al. (2011b) does not hold for this numerical example.

3.4 Relationship between $\mathcal{D}$-invariance and dDDE stability

In this subsection we aim to complement the overview of the necessary and sufficient conditions with a theoretical result which establishes a link between the stability in presence of *time-varying delay* and the existence of $\mathcal{D}$-invariant sets. This result is outlined in the following theorem which is stated without proof, for brevity.

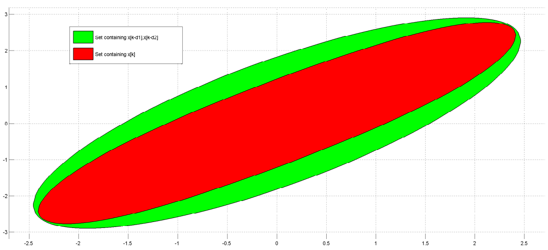

Fig. 1. $\mathcal{D}$-contractive set for the dDDE (9) with A_0, A_1 in (16)

Theorem 12. The dDDE:

$$x(k+1) = A_0 x(k-d_0) + A_1 x(k-d_1) \tag{17}$$

admits a proper $\mathcal{D}$-invariant set if and only if the time-varying dDDE

$$x(k+1) = A_0 x(k-d_0(k)) + A_1 x(k-d_1(k)) \tag{18}$$

is delay-independent stable[2].

Proposition 13. If the compact set containing the origin $\mathcal{P}$ is $\mathcal{D}$-invariant, then its convex hull $Conv(\mathcal{P})$ is $\mathcal{D}$-invariant.

Proof. One can exploit the relationship:

$$A_1 Conv(\mathcal{P}) \oplus A_2 Conv(\mathcal{P}) = \tag{19}$$

$$Conv(A_1\mathcal{P}) \oplus Conv(A_2\mathcal{P}) = \tag{20}$$

$$Conv(A_1\mathcal{P} \oplus A_2\mathcal{P}) \tag{21}$$

The first equality is straightforward. For the second one, let $\mathcal{P}_1, \mathcal{P}_2 \subset \mathbb{R}^n$, and let $x \in Conv(\mathcal{P}_1 \oplus \mathcal{P}_2)$, then $x = \sum \lambda_i(x_i + y_i)$ with $x_i \in \mathcal{P}_1$ and $y_i \in \mathcal{P}_2$, $\lambda_i \geq 0$ and $\sum \lambda_i = 1$, then $x = \sum \lambda_i x_i + \sum \lambda_i y_i \in Conv(\mathcal{P}_1) \oplus Conv(\mathcal{P}_2)$. Suppose now that $x \in Conv(\mathcal{P}_1) \oplus Conv(\mathcal{P}_2)$ then $x = \sum \lambda_i x_i + \sum \beta_j y_j$,with $\sum \lambda_i = \sum \beta_j = 1$, and $\lambda_i, \beta_j \geq 0$, $x_i \in \mathcal{P}_1, y_j \in \mathcal{P}_2$. since $\sum \lambda_i \sum \beta_j = \sum_{i,j} \lambda_i \beta_j = 1$ we can write $x = \sum_{i,j} \lambda_i \beta_j (x_i + y_j)$, then $x \in Conv(\mathcal{P}_1 \oplus \mathcal{P}_2)$. Note that

$$A_1\mathcal{P} \oplus A_2\mathcal{P} \subset \mathcal{P} \implies Conv(A_1\mathcal{P} \oplus A_2\mathcal{P}) \subset Conv(\mathcal{P}) \tag{22}$$

to conclude that:

$$A_1 Conv(\mathcal{P}) \oplus A_2 Conv(\mathcal{P}) \subset Conv(\mathcal{P}) \tag{23}$$

□

Remark 14. Property iv of Proposition 7 raises a warning on the convex hull (with two or more operands) which is not a closed operation over the class of $\mathcal{D}$-invariant sets. However, Proposition 13 points out that for one $\mathcal{D}$-invariant operand, the convex hull operation preserves $\mathcal{D}$-invariance.

4. CONSTRUCTION OF $\mathcal{D}$-INVARIANT SETS BASED ON SET ITERATIONS

Supposing that (1) admits a $\mathcal{D}$-invariant set, we address now the construction procedures. We use the fact that the existence of a $\mathcal{D}$-invariant set is exactly equivalent, by Lemma 6, to the verification of $A_0\mathcal{P} \oplus A_d\mathcal{P} \subseteq \mathcal{P}$. To simplify the explanation, we first define the forward mapping :

$$\begin{aligned} \Phi &: Com(\mathbb{R}^n) \to Com(\mathbb{R}^n) \\ &\Phi(\mathcal{P}) = A_0\mathcal{P} \oplus A_d\mathcal{P}; \end{aligned} \tag{24}$$

and the mapping based on the union:

$$\begin{aligned} \Psi &: Com(\mathbb{R}^n) \to Com(\mathbb{R}^n) \\ &\Psi(\mathcal{P}) = \bigcup(\mathcal{P}, A_0\mathcal{P} \oplus A_d\mathcal{P}) = \bigcup(\mathcal{P}, \Phi(\mathcal{P})). \end{aligned} \tag{25}$$

Note that even if $\mathcal{P}$ is convex, $\Psi(\mathcal{P})$ is not necessarily convex.

Remark 15. We enumerate here some useful properties of the mappings defined in (24-25):

- S1 If a given set $\mathcal{P}$ (convex or not) is $\mathcal{D}$-invariant for (1), then $\Phi(\mathcal{P}) \subseteq \mathcal{P}$.
- S2 k-iterates over the family of sets is set-wise non decreasing $\Psi^{k-1}(\mathcal{P}) \subseteq \Psi^k(\mathcal{P}), \forall k \geq 1$ with $\Psi^k(\mathcal{P}) = \Psi(\Psi^{k-1}(\mathcal{P}))$ for $k > 0$ and $\Psi^0(\mathcal{P}) = \mathcal{P}$.
- S3 If $\mathcal{P}$ is $\mathcal{D}$-invariant for (1) then $\Phi^k(\mathcal{P})$ is set-wise non increasing $\Phi^k(\mathcal{P}) \subseteq \Phi^{k-1}(\mathcal{P}), \forall k \geq 1$.

4.1 Basic set-iterates procedure for the construction of $\mathcal{D}$-invariant sets

We describe in this part the basic steps of an iterative construction of $\mathcal{D}$-invariant sets. Under the assumption that such an invariant set exists for the system (1), we can always scale it using property "i" of Proposition 7 such that it encompasses the initial set $\mathcal{Q}$.

This algorithm considers as an input argument an arbitrary bounded set $\mathcal{Q}$ containing the origin.

Algorithm 1. Basic (non-convex) set-iterates procedure

Input: A bounded set $\mathcal{Q} \in \mathbb{R}^n$ containing the origin; the matrices $A_0, A_d \in \mathbb{R}^{n\times n}$ describing the system (1)

Output: $\mathcal{R}$ a $\mathcal{D}$-invariant set

$\mathcal{R}_0 = \mathcal{Q}$;
$\mathcal{R}_1 = \Phi(\mathcal{Q}) = A_0\mathcal{Q} \oplus A_d\mathcal{Q}$;
$i = 1$;
while $\mathcal{R}_i \not\subset \mathcal{R}_{i-1}$ **do**
  $\mathcal{R}_{i+1} = \Psi(\mathcal{R}_i) = \bigcup(\mathcal{R}_i, A_0\mathcal{R}_i \oplus A_d\mathcal{R}_i)$;
  $i = i + 1$;
end

return$\mathcal{R} = Conv(\mathcal{R}_i)$

Remark 16. If there exists a $\mathcal{D}$-invariant set for the system (1), then Algorithm 1 constructs a non-decreasing sequence which converges to a $\mathcal{D}$-invariant set. The finite determinedness is related to the asymptotic stability of the system (1).

Note that the iterations and the limit set are non-convex and this is related to the union operation in $\Psi(\cdot)$. Since the intersection of $\mathcal{D}$-invariant sets is also $\mathcal{D}$-invariant (see property ii of Proposition 7), the sequence of Algorithm 1 converges toward the closest, in the sense of Hausdorff distance, $\mathcal{D}$-invariant superset.

Example 17. Let us consider the following dynamical system:

$$x(k+1) = \begin{bmatrix} 0.1 & 0 \\ 0.4 & 0.1 \end{bmatrix} x(k) + \begin{bmatrix} 0.1 & -0.2 \\ 0.4 & 0.5 \end{bmatrix} x(k-d), \tag{26}$$

[2] See Stankovic et al. (2014) for the formal definition of *delay-independent stability*.

Consider the initialization set $\mathcal{Q}$ as the ∞-norm unit circle in $\mathbb{R}^2$. A non-convex $\mathcal{D}$-invariant set is obtained iteratively by applying Algorithm 1 with 4 iterations.

Figure 2 presents this invariant set (the left one), and the image (the right one) of this set by the forward mapping $\Phi(\cdot)$. Their superposition (inclusion) shows that the non-convex set obtained is $\mathcal{D}$-invariant.

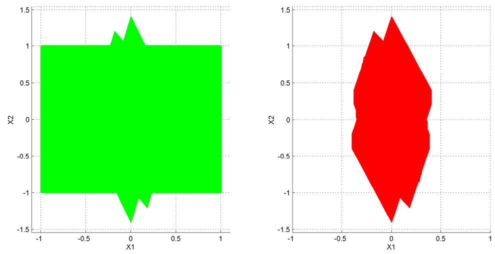

Fig. 2. Graphical illustration of $\mathcal{D}$-invariance for a non-convex set. The $\mathcal{D}$-invariant set $\mathcal{P}$ in green(left); the set $A_0\mathcal{P} \oplus A_d\mathcal{P}$ in red(right)

Figure 3 presents the Convex hull of the obtained non-convex $\mathcal{D}$-invariant set and shows that it is $\mathcal{D}$-invariant as theoretically proved by Proposition 13.

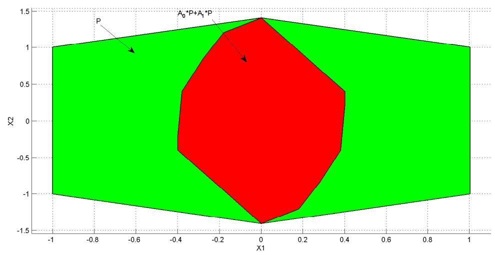

Fig. 3. Graphical illustration of the convex $\mathcal{D}$-invariant set. The $\mathcal{D}$-invariant set–green; the set $A_0\mathcal{P} \oplus A_d\mathcal{P}$ –red

4.2 Convex set-iterates procedure for the construction of $\mathcal{D}$-invariant sets

We describe briefly in this part the main steps of an iterative construction of $\mathcal{D}$-invariant sets while manipulating only convex sets. This algorithmic routine was proposed by Lombardi et al. (2011b), but we recall it here in light of Theorem 12 and Algorithm 1. Let us define the two mappings :

$$\begin{aligned} \Omega : ComC(\mathbb{R}^n) &\rightarrow ComC(\mathbb{R}^n) \\ \Omega(\mathcal{P}) &= A_0\mathcal{P} \oplus A_d\mathcal{P}; \end{aligned} \tag{27}$$

$$\begin{aligned} \Xi : ComC(\mathbb{R}^n) &\rightarrow ComC(\mathbb{R}^n) \\ \Xi(\mathcal{P}) &= Conv(\mathcal{P}, A_0\mathcal{P} \oplus A_d\mathcal{P}) = Conv(\mathcal{P}, \Omega(\mathcal{P})). \end{aligned} \tag{28}$$

Given a convex set $\mathcal{P} \in ComC(\mathcal{R}^n)$, the sequence $\Xi^k(\mathcal{P}), k > 0$ converges toward a convex $\mathcal{D}$-invariant set (Lombardi et al. (2011b)). The main objective of this procedure remains the same as the previous one: enlarge the candidate set via the Convex hull operation, by exploiting its inclusion in a $\mathcal{D}$-invariant superset.

Algorithm 2. Convex set-iterates converging to a $\mathcal{D}$-invariant set

Input: A convex set $\mathcal{Q} \in \mathbb{R}^n$ containing the origin in the interior; the matrices $A_0, A_d \in \mathbb{R}^{n\times n}$

Output: $\mathcal{R}$ Convex $\mathcal{D}$-invariant set

$\mathcal{R}_0 = \mathcal{Q}$;
$\mathcal{R}_1 = \Omega(\mathcal{Q}) = A_0\mathcal{Q} \oplus A_1\mathcal{Q}$;
$i = 1$;
while $\mathcal{R}_i \not\subset \mathcal{R}_{i-1}$ **do**
  $\mathcal{R}_{i+1} = \Xi(\mathcal{R}_i) = Co(\mathcal{R}_i, A_0\mathcal{R}_i \oplus A_d\mathcal{R}_i)$;
  $i = i + 1$;
end

return$\mathcal{R} = \mathcal{R}_i$

This algorithm, unlike the previous one, manipulates convex sets with all their computational advantages. In each iteration, the convex hull of the present set and the forward mapping of the same set $\mathcal{P}_i$ is obtained. In comparison with Algorithm 1, the main objective is to enlarge the set $\mathcal{P}_i$ in each iteration, without checking if the set in question is convex or not. The common objective is to obtain a $\mathcal{D}$-invariant set and exploit Proposition 13 which guarantees that the convex hull of this set is also $\mathcal{D}$-invariant. This characteristic can be very interesting from the computational point of view since the iteration avoids the enumeration of the convex sub-sets defining the non-convex regions.

4.3 Complexity and speed of convergence

In this section, we point to the possible extension of Algorithms 1-2 in order to improve the convergence speed. Instead of performing one forward mapping in each iteration before checking $\mathcal{D}$-invariance, N forward mappings are performed in each iteration. This alternative can offer a compromise between the complexity of the intermediary sets and the number of iterations.

Algorithm 3. Auxiliary set-iterates procedure

Input: A bounded convex set containing the origin $\mathcal{Q} \in \mathbb{R}^n$; the matrices $A_0, A_d \in \mathbb{R}^{n\times n}$;$N$ the number of forward mappings in one iteration

Output: $\mathcal{R}$ Convex $\mathcal{D}$-invariant set

$\mathcal{R}_0 = \mathcal{Q}$;
$\mathcal{R}_1 = \Omega(\mathcal{Q}) = A_0\mathcal{Q} \oplus A_1\mathcal{Q}$;
$Aux_1 = \mathcal{R}_0$;
$i = 1$;
while $\mathcal{R}_i \not\subset \mathcal{R}_{i-1}$ **do**
  for $m = 1 : N$ **do**
    $Aux_{m+1} = \Phi(Aux_m)$
  end
$Aux = [Aux_1, Aux_2, \ldots, Aux_{N+1}]$;
$\mathcal{R}_i = Conv(Aux)$;
$\mathcal{R}_{i+1} = \Omega(\mathcal{R}_i)$;
$i = i + 1$;
$Aux_1 = \mathcal{R}_i$;
end

return$\mathcal{R} = \mathcal{R}_i$

Example 18. Let us consider the following dynamical system :

$$x(k+1) = \begin{bmatrix} 0.2 & 0.1 \\ 0 & 0.6 \end{bmatrix} x(k) + \begin{bmatrix} 0.5 & 0 \\ 0.1 & 0.3 \end{bmatrix} x(k-d). \quad (29)$$

Let

$$\text{Q}= \left\{ x \in \mathbb{R}^2 | \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ -\sqrt{2} & \sqrt{2} \\ \sqrt{2} & \sqrt{2} \\ -\sqrt{2} & -\sqrt{2} \end{bmatrix} x \leq \begin{bmatrix} 0.5 \\ 1 \\ 0.5 \\ 1 \end{bmatrix} \right\}$$

be the initialization set. By applying on one hand Algorithm 3 with $N = 2$ and on the other hand Algorithm 2, two different $\mathcal{D}$-invariant sets are obtained for the dynamical system (29) in $2 * (N = 2)$ and 18 iterations, respectively. Figure 4 presents these sets.

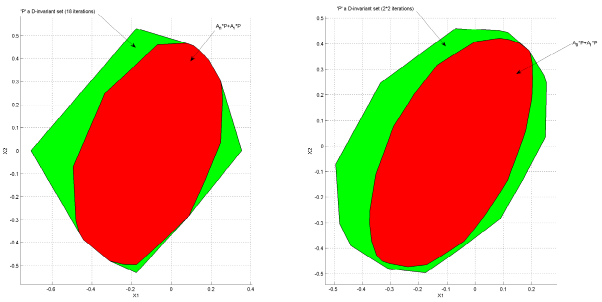

Fig. 4. Graphical illustration of $\mathcal{D}$-invariant sets obtained by Algorithm 2 (left) and Algorithm 3 (right)

It becomes clear that under the assumption that a $\mathcal{D}$-invariant set exists, a construction procedure exists. More than that, one can also use the algorithmic construction (Algorithm 2) as an induced tool to check if a $\mathcal{D}$-invariant set can/cannot be obtained, whenever the dDDE satisfies the necessary conditions for the existence of such invariant sets. To illustrate this idea, Example 10, which raises a doubt about the sufficiency of the matrix-pencil based conditions (Stankovic et al. (2014)), will be discussed in the sequel. By computing the set iterations up to strict inclusion into the initial one, convergence/divergence can be inferred. If the initial set $\mathcal{Q}$ for Algorithm 2 is the ∞-norm unit circle in $\mathbb{R}^2$ and the dDDE is given by the matrices in Example 10, then

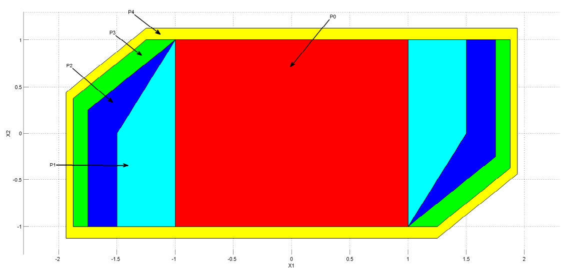

Fig. 5. Sequence of the forward mappings $Conv(\mathcal{P}, A_0\mathcal{P} \oplus A_d\mathcal{P})$ leading to a strict superset in 4 iterations

after 4 iterations one obtains the sequence in Figure 5. The set iteration can be stopped as long as $\mathcal{Q}$ is a strict subset of $\mathcal{P}_4$. This represents a proof by construction that forward set iterations diverge and the system does not admit a $\mathcal{D}$-invariant set.

5. CONCLUSION

This paper discusses the positive invariance for discrete time-delay systems. Necessary or sufficient conditions for the existence of $\mathcal{D}$-*invariant* sets have been gathered and discussed. The relationship between $\mathcal{D}$-*invariance* and stability was studied for discrete delay difference equations (dDDEs). The construction of $\mathcal{D}$-invariant sets via set iterations was shown to benefit from the convexification despite the fact that set forward mappings based on the original dDDE lead to a non-convex $\mathcal{D}$-invariant set.

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