Utilizing Reachability Analysis in Point Location Problems

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***Abstract*— Recent results in parametric mathematical programming established that the explicit piecewise affine solution to some optimal control problems can be defined on a partition of the state space. An optimal feedback control law is associated with each member of the state space partition. Instead of solving an optimization problem online at every time instant, the recent, so–called explicit, MPC schemes require merely the identification of the member of the solution partition that contains the measured state. Once this set is found, the affine optimal control law associated with the set is evaluated and applied to the system. This set–membership problem is referred to as the point location problem. We demonstrate that, under mild assumptions, the point location problem only needs to be solved at the initialization of the control scheme rather than at every time instant. Given a member of the solution (state-space) partition, the system dynamics and the explicit control law are utilized to obtain the set of states that can be reached at the next time instant. The members of the partition intersecting this one-step forward reach set are, under our assumptions, guaranteed to contain the process state at next point in time. A direct consequence of the presented results is the fact that it is sufficient to solve a reduced point location problem instead of resolving the entire point location problem at every time instant.**

***Index Terms*— Parametric programming, Explicit model predictive control, Piecewise affine functions.**

I. Problem Setup

Consider the following system:

$$x^+ = f(x, u, w) \tag{1}$$

where $(x, u) \in \mathcal{Y} \subset \mathbb{R}^n \times \mathbb{R}^m$ is the state and input, $x^+ \in \mathbb{R}^n$ is the successor state, and $w \in \mathcal{W} \subset \mathbb{R}^p$ is a disturbance. In this note $f(\cdot)$ can take the form $Ax + Bu$, $Ax + Bu + w$, or $Ax + Bu + w$ with $[A\,B] \in \mathbb{M} := \operatorname{conv}\left([A^1\,B^1], [A^2\,B^2], \ldots, [A^J\,B^J]\right)$. We assume that there exists a control law $u^* : \mathbb{R}^n \to \mathbb{R}^m$ that is a solution to some optimal control problem $\mathbb{P}(\cdot)$ such that $u^*(\cdot)$ is piecewise affine on a polyhedral cover $\mathcal{R} := \{R^1, R^2, \ldots, R^k\}$ of the set of states $\mathcal{X}$ for which the optimal control problem has a solution. This is the case for some explicit approaches to MPC problems, see e.g. [1]–[3]. The explicit version of MPC [1], [2] can be summarized as follows:

1) Measure the current state of the system.
2) Identify the the region $R^i \in \mathcal{R}$ that contains the current state (this will be referred to as the point location problem).
3) Apply the control associated with the region identified in step 2) to the plant.
4) Return to step 1).

In this note we aim at reducing the computational effort needed in point 2), this is treated in Section III after the point location problem has been formalized in Section II.

In the sequel, we will use the following notation: Given two sets $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^n$, the Minkowski set addition is defined as $X \oplus Y := \{x + y \mid x \in X, y \in Y\}$.

II. Point Location Problem

The point location problem can be stated as: Given a polyhedral cover $\mathcal{P} := \{P^1, P^2, \ldots, P^K\}$ of the polyhedron P, such that for every pair $(P^i, P^j) \in \mathcal{P} \times \mathcal{P}$, $i \neq j$, we have $\operatorname{int}\left(P^i\right) \cap \operatorname{int}\left(P^j\right) = \emptyset$, and a point $x \in P$, find $P^i \in \mathcal{P}$ such that $x \in P^i$.

There exist several approaches for the purpose of solving the point location problem, and the reader is referred to the respective paper for details:

1) Linear search through $\mathcal{P}$.
2) Comparison of value functions [4].
3) Using a binary search tree [5].
4) Logarithmic time approach [6].

Instead of searching through the entire set $\mathcal{P}$ at each sample one can utilize reachability analysis to reduce the number of polyhedral sets that are candidates to contain the state at the next time instant. This reachability approach is proposed in the next section.

III. Utilizing Reachability Analysis in Point Location Problems

Under the assumptions on $f(\cdot)$ and $u^*(\cdot)$, it is possible to compute a convex outer approximation of the one-step forward reach set associated with a set $R \in \mathcal{R}$, that is, a convex outer approximation of the set

$$R^+ := \{f(x, u^*|_R(x), w) \,|\, x \in R, w \in \mathcal{W}\},$$

where the control law $u^*(\cdot)$ is defined as

$$u^*(x) = u^*|_R(x) := K_R x + k_R \quad \text{if} \quad x \in R,\ R \in \mathcal{R}.$$

Given a set $R \in \mathcal{R}$ we can map the set R one-step forward in time yielding the set R^+. We associate with this reach set the subset $\mathcal{M}(R^+)$ of $\mathcal{R}$ defined by $\left\{R^i \in \mathcal{R} \,\middle|\, R^+ \cap R^i \neq \emptyset\right\}$ and define a new set of polyhedra $\mathcal{N}(R^+) := \left\{R^i \cap R^+ \,\middle|\, R^i \in \mathcal{M}(R^+)\right\}$. Thus, if our previous state was contained in R, then the point location problem reduces to a search through set $\mathcal{N}(R^+)$. The off-line computation required and the online algorithm are stated in Algorithm 1 and 2, respectively.

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Algorithm 1 Offline computation.

Input: The system $x^+ = f(x,u,w)$, the constraints set $(\mathcal{Y}, \mathcal{W})$, and an explicit control law $u^*(\cdot)$.
Output: A collection $\mathcal{C}$ of pairs $(R, \mathcal{N}(R^+))$.
1: $\mathcal{C} \leftarrow \emptyset$.
2: **for** all $R \in R$ **do**
3: Compute $R^+ := \{f(x, u^*|_R(x), w)\,|x \in R, w \in \mathcal{W}\}$.
4: Compute $\mathcal{N}(R^+) := \left\{R^i \in \mathcal{R} \,\middle|\, R^+ \cap R^i \neq \emptyset\right\}$.
5: $\mathcal{C} \leftarrow \mathcal{C} \cup \{(R, \mathcal{N}(R^+))\}$.
6: **end for**

Algorithm 2 Explicit MPC utilizing reachability.

Input: An explicit control law $u^*(\cdot)$ defined on $\mathcal{R}$, and the output $\mathcal{C}$ of Algorithm 1.
1: **Initialization:** Given the measured state x, solve the point location problem for $\mathcal{R}$, i.e. determine R such that $x \in R$. Apply $u^*|_R(x)$ to the plant.
2: **while** MPC algorithm is running **do**
3: Measure the state x.
4: Given R, recall $\mathcal{N}(R^+)$ from $\mathcal{C}$ and solve the point location problem for $\mathcal{N}(R^+)$, that is, find $\bar{R} \in \mathcal{N}(R^+)$ such that $x \in \bar{R}$.
5: Apply the control $u^*|_{\bar{R}}(x)$ to the plant.
6: $R \leftarrow \bar{R}$.
7: **end while**

A. Evaluation of piecewise affine control laws for the regulation problem of deterministic systems

Consider the class of systems where

$$x^+ = f(x,u,0) = Ax + Bu.$$

The reach set R^+ associated with R is easily found from

$$\begin{aligned} R^+ : &= \{Ax + Bu^*|_R(x)\,|x \in R\} \\ &= (A + BK_R)R \oplus \{Bk_R\}. \end{aligned}$$

We see that if we use the Algorithms 1 and 2 on this problem, it resembles open loop control in the sense that we have assumed that our system is completely deterministic. To remedy this we consider the case where measurement noise and input corruption are represented by the model (model uncertainty is treated in the next subsection):

$$x^+ = Ax + Bu + w,\ w \in \mathcal{W}.$$

In this case the reach set becomes:

$$\begin{aligned} R^+ : &= \{Ax + Bu^*|_R(x) + w\,|x \in R, w \in \mathcal{W}\} \\ &= (A + BK_R)R \oplus \mathcal{W} \oplus \{Bk_R\}. \end{aligned}$$

B. Evaluation of piecewise affine control laws for the regulation problem of uncertain systems

Consider the class of systems where

$$\begin{aligned} x^+ &= f(x,u,w) = Ax + Bu + w, \\ \begin{bmatrix} A & B \end{bmatrix} &\in \mathbb{M} := \operatorname{conv}\left(\begin{bmatrix} A^1 & B^1 \end{bmatrix}, \ldots, \begin{bmatrix} A^J & B^J \end{bmatrix}\right). \end{aligned}$$

The reach set associated with $R \in \mathcal{R}$ is given by

$$\begin{aligned} R^+ : &= \left\{ Ax + Bu^*|_R(x) + w \;\middle|\; \begin{array}{l} x \in R \\ \begin{bmatrix} A & B \end{bmatrix} \in \mathbb{M} \\ w \in \mathcal{W} \end{array} \right\} \\ &= \bigcup_{x \in R} \left\{ Ax + Bu^*|_R(x) + w \;\middle|\; \begin{array}{l} \begin{bmatrix} A & B \end{bmatrix} \in \mathbb{M} \\ w \in \mathcal{W} \end{array} \right\}. \end{aligned}$$

Noting that this set is generally non-convex, we use the convexification of R^+, i.e. $\bar{R}^+ := \operatorname{conv}(R^+)$ or any other suitable convex outer approximation. Noting that for each fixed pair $\begin{bmatrix} A & B \end{bmatrix} \in \mathbb{M}$ we have that $\{Ax + Bu^*|_R(x) + w\,|x \in R, w \in \mathcal{W}\}$ is a polyhedron, hence $\bar{R}_+$ is also a polyhedron. It is straightforward to use Algorithm 1 and 2 in the parameter uncertain case if step 3 of Algorithm 1 is replaced by: Compute $\bar{R}^+$.

C. Evaluation of piecewise affine control laws for set-point tracking for deterministic systems

The tracking problem is slightly different than the regulation, and for brevity we only consider the deterministic system on the form

$$x^+ = f(x,u,0) = Ax + Bu.$$

The objective is to minimize deviation from some desired reference signal s. In this case we assume that $s \in S$, where S is a polyhedron, but that s can change arbitrarily from one sample to the next. The optimal control law is then a function of the set-point and initial state:

$$u^*(x,s) = u^*|_R(x,s) := K_R^1 x + K_R^2 s + k_R \text{ if } (x,s) \in R,$$

which is defined on a polyhedral cover $\mathcal{R}$ of the set $\mathcal{X} \times S$.

The reach set is, in this case, given by:

$$\begin{aligned} R^+ &: = R_X^+ \times S, \text{ where} \\ R_X^+ &: = \{Ax + Bu^*|_R(x,s)\,|(x,s) \in R\}. \end{aligned}$$

IV. Conclusion

The reachability approach, under certain mild assumptions on the problem data, reduces the online computational effort needed to solve the the point location problem. This reduction naturally comes at the expense of increased offline computational effort and required storage space.

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