Robust Adaptive Control of Minimum Phase Nonlinear Systems[1]

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Abstract

We show that an adaptive input/output feedback linearization control scheme for minimum phase nonlinear systems is robust with respect to unstructured plant uncertainties that include such as unmodeled dynamics and disturbances, provided the adaptive law is modified in the same fashion as for linear systems, and the plant nonlinearities satisfies some growth constraints. In the analysis we utilize weighted $\mathcal{L}_2$-norms analogously to the case with a linear model.

Keywords: Adaptive Control, Nonlinear Systems, Robustness, Stability, Feedback Linearization.

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1 Introduction

For adaptive control of linear systems, the robustness issue has turned out to be very important, since it has been demonstrated that even small modeling errors or disturbances can lead to instability if an adaptive controller designed for the ideal case is used [7, 10]. Several modifications of the standard adaptive algorithms have been proved to improve the robustness [7, 11, 20, 13, 26], and a unified theory based on separating the design of the parameter estimator and controller, and combining them using the certainty equivalence principle has been developed [12, 32, 9].

In recent years, there have been considerable interest in adaptive control of non-linear systems using different variations of feedback linearization [22, 28, 31, 4, 27, 19, 17, 30]. The common assumptions are that the full state is measurable, the system is exactly linearly parameterized and feedback linearizable (input/state or input/output). With a few exceptions, the robustness issue is not addressed. In [31, 16] it is shown, using singular perturbation theory, that the adaptive scheme is robust when it is designed based on a reduced model. An analysis that is similar in spirit to the one presented here is given in [34], but again only with structured uncertainty (unmodeled dynamics), is considered. An alternative way to achieve robustness is to use a sliding controller in addition to the adaptive controller, as discussed in [30, 25]. The aim of the present paper is to perform a robustness analysis of a modified non-linear adaptive controller based on [28] using the tools that have been useful for the analysis in the linear case [32, 9], namely input/output stability theory, in particular weighted $\mathcal{L}_2$-norms. Small unstructured disturbances, modeling error and unmodeled dynamics are allowed.

The paper is organized as follows: After defining the notation, we review the basics of input/output feedback linearization and present the control structure chosen for this study. In section 4 we review some results on parameter estimators, and in section 5 we discuss the stability of the adaptive system. Finally, we discuss the assumptions made, illustrate the theory with a simple example, and make some concluding remarks.

2 Notation

The Euclidean norm of a vector $x \in R^n$ is denoted $||x|| = \sqrt{x^T x}$. The norm of a matrix A is $||A|| = \sup_{||x||=1} ||Ax||$. The truncation of the signal $x : [0, \infty) \to R^n$ is $x_t : [0, \infty) \to R^n$, defined by

$$x_t(\tau) = \begin{cases} x(\tau) & \text{if } \tau \le t \\ 0 & \text{if } \tau > t \end{cases}$$

We define the so-called weighted $\mathcal{L}_2$-norm, or $\mathcal{L}_{2e}$-norm, of $x \in \mathcal{L}_{2e}$ by

$$||x_t||_2 = \sqrt{\int_0^t x^T(\tau) x(\tau) d\tau}$$

and the $\mathcal{L}_{2e}^\delta$-norm of $x \in \mathcal{L}_{2e}^\delta$ by

$$||x_t||_2^\delta = \sqrt{\int_0^t e^{-\delta(t-\tau)} x^T(\tau) x(\tau) d\tau} \tag{1}$$

for some real $\delta \ge 0$. The H_∞-norm of a stable, proper transfer function matrix $H(s)$ (where s is the Laplace operator) is defined as

$$||H(s)||_\infty = \operatorname{ess\,sup}_\omega \overline{\sigma}(H(j\omega))$$

where $\overline{\sigma}(A)$ denotes the largest singular value of the matrix A. We also define for some real $\delta \ge 0$

$$||H(s)||_\infty^\delta = ||H(s - \delta/2)||_\infty$$

The symbol K denotes a generic finite positive constant.

For some signal $y : [0,\infty) \to [0,\infty)$, we define the family of y-small in mean vector-signals as

$$S(y) \;=\; \left\{ x : [0,\infty) \to R^n \;\middle|\; \int_t^{t+T} x^T(\tau)x(\tau)d\tau \leq K + K\int_t^{t+T} y(\tau)d\tau \qquad \text{for all } t, T \geq 0 \right\}$$

A function $f : X \to Y$, where $X \subset R^n$ and $Y \subset R^m$, is called smooth if it is continuously differentiable a sufficient number of times, and Lipschitz on X if there exists a constant $L \geq 0$ (called Lipschitz constant) such that $||f(x_1) - f(x_2)|| \leq L||x_1 - x_2||$ for all $x_1, x_2 \in X$. The rest of the notation is standard: A review of $\mathcal{L}_p$-norms can be found in [6], and the notation and an introduction to the differential geometric topics can be found in e.g. [14, 24].

3 Feedback Linearization

Feedback linearization is a systematic control design procedure that works for a significant class of nonlinear systems, namely feedback linearizable minimum phase systems where u enters affinely in the state equation. In general, the feedback linearization procedure aims at transforming the nonlinear system into a linear one, using state-feedback and a change of coordinates. The resulting linear system can then be controlled using traditional linear controller design. After the pioneering work by Brockett [1], the exact input/output linearization problem [15] and the input/state linearization problem [8] were solved. For cases when there is parametric uncertainty in the model, adaptive versions of these control structures were proposed [28, 22, 31, 4, 27, 19, 17]. As in most adaptive systems, linear parameterization is a key assumption for establishing global stability.

Let us start with a brief review of input/output linearization. We consider the SISO system

$$\dot{x} \;=\; f(x) + g(x)u \tag{2}$$

$$y \;=\; h(x) \tag{3}$$

where $x \in X \subset R^n$ is the measured state-vector, $u \in R$ is input, and $y \in R$ is output. X is supposed to be an open connected subset of the state-space (not excluding the possibility $X = R^n$). The vector fields $f, g : X \to R^n$, and the output function $h : X \to R$ are all assumed to be smooth.

Definition 1 *The system (2)-(3) has uniform strong relative degree γ in X if*

$$\begin{aligned} L_g h(x) = L_g L_f h(x) = \cdots = L_g L_f^{\gamma-2} h(x) &\;=\; 0 \\ L_g L_f^{\gamma-1} h(x) &\;\neq\; 0 \end{aligned}$$

for all $x \in X$. □

Notice that not all systems have a well defined uniform strong relative degree, e.g. when $L_g L_f^{\gamma-1} h(x_0) = 0$ for some point $x_0 \in X$, and $L_g L_f^{\gamma-1} h(x) \neq 0$ elsewhere [5, 14]. We will not consider such systems here:

Assumption 1 *The system has uniform strong relative degree γ in X.* □

The control objective is to choose u such that the closed loop is stable and y tracks an arbitrary reference trajectory y_M that satisfies

Assumption 2 *The scalar reference trajectory y_M is smooth and bounded, i.e. $y_M, y_M^{(1)}, ..., y_M^{(\gamma)}$ are bounded.* □

Before we consider the adaptive case, let us first review non-adaptive feedback linearization, i.e. when f, g and h are known. As shown in [14], we can differentiate the output equation γ times

$$y^{(\gamma)} \;=\; L_f^{\gamma} h(x) + L_g L_f^{\gamma-1} h(x) u$$

Since $L_g L_f^{\gamma-1} h(x) \neq 0$ for all $x \in X$, the feedback

$$u = \frac{v - L_f^{\gamma} h(x)}{L_g L_f^{\gamma-1} h(x)} \tag{4}$$

with external input $v \in R$ is well defined, and yields the linearized system

$$y^{(\gamma)} = v$$

The reference trajectory can be asymptotically tracked using the linear external controller

$$v = y_M^{(\gamma)} + \alpha_{\gamma-1}\left(y_M^{(\gamma-1)} - y^{(\gamma-1)}\right) + ... + \alpha_0(y_M - y)$$

if the polynomial $s^\gamma + \alpha_{\gamma-1}s^{\gamma-1} + ... + \alpha_0$ with real coefficients $\alpha_0, \alpha_1, ..., \alpha_{\gamma-1}$ is Hurwitz. We can compute $y^{(k)} = L_f^k h(x)$ for $1 \le k \le \gamma - 1$, since the system and states are known. If the tracking error is defined as $e = y_M - y$, the exponentially stable closed loop system becomes

$$e^{(\gamma)} + \alpha_{\gamma-1}e^{(\gamma-1)} + ... + \alpha_0 e = 0$$

Next, we consider the situation when f, g and h may be unknown. Notice that since x is measured, the output function $y = h(x)$ will typically be subject to design, and will therefore be known exactly.

In [28] it is shown that if f, g and h are linearly parameterized and their exact structure is known, then an exact linearly parameterized structure for the Lie-derivatives $L_f^k h$ for $k = 1, .., \gamma$, and $L_g L_f^{\gamma-1} h$ can be found, using a reparameterization, and the tracking error satisfies

$$e^{(\gamma)} + \alpha_{\gamma-1}e^{(\gamma-1)} + ... + \alpha_0 e + \varphi^T\tilde{\theta} = 0$$

for some regressor vector φ and parameter error vector $\tilde{\theta}$. In practise, the model consisting of f, g and h will only be an approximation, i.e. $f(x) = \varphi_f(x)\theta_f + \nu_f$, where ν_f is some modeling error that depends on x and t. Using either the reparameterization approach or direct approximation of the Lie-derivatives by linearly parameterized functions, we get

$$\left.\begin{array}{rcl} L_f^\gamma h(x) &=& \varphi_{f,\gamma}^T(x)\theta_{f,\gamma} + \nu_{f,\gamma} \\ &\vdots& \\ L_f h(x) &=& \varphi_{f,1}^T(x)\theta_{f,1} + \nu_{f,1} \\ L_g L_f^{\gamma-1}h(x) &=& \varphi_{g,f}^T(x)\theta_{g,f} + \nu_{g,f} \end{array}\right\} \quad (5)$$

where $\varphi_{f,1}, ..., \varphi_{f,\gamma-1}, \varphi_{f,\gamma}, \varphi_{g,f}$ are known regressor vectors, $\theta_{f,1}, ..., \theta_{f,\gamma-1}, \theta_{f,\gamma}, \theta_{g,f}$ are unknown "true parameters", and $\nu_{f,1}, \ldots, \nu_{f,\gamma-1}, \nu_{f,\gamma}, \nu_{g,f}$ are unknown unstructured uncertainties, which incorporates modeling error, unmodeled dynamics, noise, and disturbances. Notice that the term "true parameters" calls for some clarification, since (5) hold for any choice of "true parameter" vectors. However, some choices give smaller model uncertainty than others. What we mean by "true parameters" are those parameters that minimize the average unstructured uncertainty, in some sense.

The certainty equivalence adaptive controller based on an arbitrary linearly parameterized approximate model is

$$u = \frac{v - \varphi_{f,\gamma}^T(x)\hat{\theta}_{f,\gamma}}{\varphi_{g,f}^T(x)\hat{\theta}_{g,f}} \quad (6)$$

where

$$v = y_M^{(\gamma)} + \alpha_{\gamma-1}\left(y_M^{(\gamma-1)} - \varphi_{f,\gamma-1}^T(x)\hat{\theta}_{f,\gamma-1}\right) + ... + \alpha_1\left(y_M^{(1)} - \varphi_{f,1}^T(x)\hat{\theta}_{f,1}\right) + \alpha_0(y_M - h(x)) \quad (7)$$

which gives the error equation

$$\begin{array}{l} e^{(\gamma)} + \alpha_{\gamma-1}e^{(\gamma-1)} + ... + \alpha_0 e - \left(\varphi_{g,f}^T(x)\tilde{\theta}_{g,f} - \nu_{g,f}\right)u \\ -\left(\varphi_{f,\gamma}^T(x)\tilde{\theta}_{f,\gamma} - \nu_{f,\gamma}\right) - \alpha_{\gamma-1}\left(\varphi_{f,\gamma-1}^T(x)\tilde{\theta}_{f,\gamma-1} - \nu_{f,\gamma-1}\right) - ... - \alpha_1\left(\varphi_{f,1}^T(x)\tilde{\theta}_{f,1} - \nu_{f,1}\right) = 0 \end{array}$$

where $\hat{\theta}_\star$ is a parameter estimate, and $\tilde{\theta}_\star = \hat{\theta}_\star - \theta_\star$ is parameter error. We choose the coefficients $\alpha_0, \alpha_1, ..., \alpha_{\gamma-1}$ such that the transfer function

$$M(s) = \frac{1}{s^\gamma + \alpha_{\gamma-1}s^{\gamma-1} + ... + \alpha_1 s + \alpha_0}$$

has the property that there exists a $\delta > 0$ such that $M(s)$ has no poles in the part of the complex plane where $\mathrm{Re}(s) \geq -\delta/2$. In other words, $M(s)$ is stable, with some "stability margin" δ. We rewrite the error equation as

$$e \;=\; M(s)\left[\varphi_1^T\tilde{\theta}_1 + \varphi_2^T\tilde{\theta}_2 u + \nu_1 + \nu_2 u\right] \tag{8}$$

where

$$\varphi_1(x) = \begin{pmatrix} \varphi_{f,\gamma}(x) \\ \alpha_{\gamma-1}\varphi_{f,\gamma-1}(x) \\ \vdots \\ \alpha_1\varphi_{f,1}(x) \end{pmatrix}, \qquad \tilde{\theta}_1 = \begin{pmatrix} \tilde{\theta}_{f,\gamma} \\ \tilde{\theta}_{f,\gamma-1} \\ \vdots \\ \tilde{\theta}_{f,1} \end{pmatrix}, \qquad \nu_1 = -(\nu_{f,\gamma} + \alpha_{\gamma-1}\nu_{f,\gamma-1} + \ldots + \alpha_1\nu_{f,1})$$

and

$$\varphi_2(x) = \varphi_{g,f}(x), \qquad \tilde{\theta}_2 = \tilde{\theta}_{g,f}, \qquad \nu_2 = -\nu_{g,f}$$

In (8) we have used the standard convention that the mixed time/frequency-domain notation $y = H(s)[u]$, where y and u are time-domain signals and $H(s)$ is a transfer function, means the convolution $y = h \star u$, where $h(t)$ is the inverse Laplace transform of $H(s)$. The exponentially decaying term due to possibly non-zero initial conditions is neglected, as it does not affect the stability of the system. However, it affects the transient performance, and should be taken into account in a performance analysis.

We define

$$\nu = \nu_1 + \nu_2 u, \quad \tilde{\theta}^T = (\tilde{\theta}_1^T, \ \ \tilde{\theta}_2^T), \quad \varphi^T = (\varphi_1^T, \ \ \varphi_2^T u)$$

and get the following compact expression for the error equation

$$e \;=\; M(s)[\varphi^T\tilde{\theta} + \nu]$$

4 Parameter estimation

In this section we consider the problem of estimating on-line the unknown time-invariant parameter vector θ. For this purpose we formulate a linear parametric model [9]. From (5) we define

$$\begin{aligned} z_1 &= sW(s)[y] = W(s)\left[\varphi_{f,1}^T\theta_{f,1} + \nu_{f,1}\right] \\ &\vdots \\ z_{\gamma-1} &= s^{\gamma-1}W(s)[y] = W(s)\left[\varphi_{f,\gamma-1}^T\theta_{f,\gamma-1} + \nu_{f,\gamma-1}\right] \\ z_\gamma &= s^\gamma W(s)[y] = W(s)\left[\varphi_{f,\gamma}^T\theta_{f,\gamma} + \nu_{f,\gamma} + \varphi_{g,f}^T\theta_{g,f}u + \nu_{g,f}u\right] \end{aligned}$$

where $W(s)$ is an arbitrary stable transfer function of relative degree γ, e.g.

$$W(s) = \frac{1}{a_\gamma s^\gamma + \ldots + a_2 s^2 + a_1 s + 1} \tag{9}$$

We define

$$\begin{aligned} z &= \alpha_1 z_1 + \alpha_2 z_2 + \ldots + \alpha_{\gamma-1}z_{\gamma-1} + z_\gamma \\ &= \left(\alpha_1 s + \ldots + \alpha_{\gamma-1}s^{\gamma-1} + s^\gamma\right)W(s)[y] \\ &= W(s)\left[\varphi_1^T\theta_1 + \varphi_2^T\theta_2 u - \nu_1 - \nu_2 u\right] \\ &= W(s)\left[\varphi_1^T\right]\theta_1 + W(s)\left[\varphi_2^T u\right]\theta_2 - W(s)[\nu] \\ &= \zeta_1^T\theta_1 + \zeta_2^T\theta_2 - \eta \end{aligned} \tag{10}$$

where $\zeta_1 = W(s)[\varphi_1], \zeta_2 = W(s)[\varphi_2 u]$, and $\eta = W(s)[\nu]$. Defining $\zeta^T = (\zeta_1^T, \ \ \zeta_2^T)$, we get

$$z = \zeta^T\theta - \eta \tag{11}$$

which suggests the estimation model

$$\hat{z} = \zeta^T \hat{\theta}$$

giving the prediction error

$$\varepsilon_1 = \hat{z} - z = \zeta^T \tilde{\theta} + \eta \tag{12}$$

Eq. (11) defines a linear parametric model, where z can be computed using (10) since $(\alpha_1 s + ... + \alpha_{\gamma-1} s^{\gamma-1} + s^\gamma) W(s)$ is a proper transfer function, and ζ is computed by filtering the regressor φ using the stable filter $W(s)$. Since the unmodeled dynamics can make the uncertainty η unbounded, we introduce a scalar normalizing signal $m_s \geq 0$ [26, 9] and a signal $m > 0$ such that $m^2 = k_1 + k_2 m_s^2$ and

$$\frac{\zeta}{m}, \frac{\eta}{m} \in \mathcal{L}_\infty \tag{13}$$

for some finite constants $k_1 > 0$ and $k_2 > 0$. The normalized prediction error is then

$$\varepsilon = \frac{\varepsilon_1}{m^2} = \frac{\zeta^T \hat{\theta} - z}{m^2} \tag{14}$$

There are several parameter estimation algorithms available that give adequate parameter estimates. Here, we will only consider a modified gradient method, but also SPR/Lyapunov-design and the least squares algorithm will give similar properties of the estimator [9]. The modified gradient adaptive law is given by

$$\dot{\hat{\theta}} = -\Gamma \zeta d(\varepsilon) - w \Gamma \hat{\theta} \tag{15}$$

where $\hat{\theta}(0)$ is given, $\Gamma = \Gamma^T$ is a scaling matrix, w is a signal to be designed [9], and $d : R \to R$ is a continuous function to be designed. A standard gradient algorithm will have the form $\dot{\hat{\theta}} = -\Gamma \zeta \varepsilon$. The modifications are introduced to improve the robustness of this algorithm, in particular to avoid parameter estimate drift and instability, which is known to be a serious problem if $\eta \neq 0$. Although there are other choices available, like parameter projection into some convex compact subset of the parameter space [21], we will here only consider the following choices for w and d:

1. *Relative dead-zone* [20]: $w(t) = 0$ and

$$d(\varepsilon) = \begin{cases} \varepsilon + d_0/m, & \text{if } \varepsilon < -d_0/m \\ 0, & \text{if } |\varepsilon| \leq d_0/m \\ \varepsilon - d_0/m, & \text{if } \varepsilon > d_0/m \end{cases}$$

   where $d_0 \geq |\eta/m| > 0$ is a constant defining the relative dead-zone.

2. *σ-modification* [11]: $w(t) = \sigma$, where $\sigma > 0$ is a constant, and $d(\varepsilon) = \varepsilon$.

3. *ϵ-modification* [23]: $w(t) = \rho|\epsilon m|$, where $\rho > 0$ is a constant, and $d(\varepsilon) = \varepsilon$.

Theorem 1 *Consider the adaptive law (15) with either relative dead-zone, σ-modification, or ϵ-modification. If the normalizing signal m satisfies (13), then*

$$\begin{aligned} \varepsilon, \varepsilon m_s, \hat{\theta}, \dot{\hat{\theta}} &\in \mathcal{L}_\infty \\ \varepsilon, \varepsilon m_s &\in S\left(c + \frac{\eta^2}{m^2}\right) \end{aligned}$$

With relative dead-zone, we have in addition $\dot{\hat{\theta}} \in \mathcal{L}_1 \cap \mathcal{L}_2$, and with σ-modification or ϵ-modification we have $\dot{\hat{\theta}} \in S\left(c + \frac{\eta^2}{m^2}\right)$. The constant $c > 0$ is given by

$$c = \begin{cases} d_0 & \textit{Relative dead-zone} \\ \sigma & \sigma - \textit{modification} \\ \rho & \epsilon - \textit{modification} \end{cases} \tag{16}$$

Proof: See [9].

Remarks: The properties of the estimator are obtained without making any assumptions about signal boundedness or persistence of excitation. In the ideal case, when $\eta = w = 0$ and $d(\varepsilon) = \varepsilon$, the estimator has in addition the property $\varepsilon, \varepsilon m_s, \dot{\hat{\theta}} \in \mathcal{L}_2$.

□

To ensure controllability of the model, we apply a parameter projection to the estimate $\hat{\theta}_2$ such that the following holds:

Assumption 3 *There exists a constant $\beta_0 > 0$ such that $\varphi_2^T(x)\hat{\theta}_2 \geq \beta_0$ for all $t \geq 0$.*

Remark. In particular, if $\varphi_2(x) = (\varphi_{2,1}(x),\ \varphi_{2,2}(x),\ \cdots, \varphi_{2,p_2}(x))^T$, and there exists a constant $\overline{\varphi} > 0$ such that $\varphi_{2,i}(x) > \overline{\varphi}$ for all $i \in \{1, 2, ..., p_2\}$ and $x \in X$, the condition can be satisfied by projecting the parameter estimates such that $\hat{\theta}_{2,i}(t) \geq \beta_0/\overline{\varphi}p_2$ at all $t \geq 0$. This will restrict the parameter estimates to a convex subset of the parameter space. As shown in [9], such a projection will not influence the properties of the estimation algorithm in Theorem 1.

□

The signal m_s is defined by

$$\dot{m}_s^2 \quad = \quad -\delta m_s^2 + |u|^2 + ||x||^2 + 1 \tag{17}$$

with $m_s(0) = 0$ and $\delta > 0$ is such that $M(s)$ has no poles in the part of the complex plane where $\mathrm{Re}(s) \geq -\delta/2$. The solution to this differential equation is

$$\begin{aligned} m_s^2(t) &= m_s^2(0)e^{-\delta t} + \int_0^t e^{-\delta(t-\tau)} \left(|u|^2 + ||x||^2 + 1\right) d\tau \\ &= (||u_t||_2^\delta)^2 + (||x_t||_2^\delta)^2 + (||1_t||_2^\delta)^2 \end{aligned} \tag{18}$$

Let us verify that this definition of m_s ensures (13). Since $W(s)$ is a strictly proper transfer function it follows from Lemma 1 in Appendix A that

$$||\zeta_1(t)|| \quad \leq \quad \frac{1}{\sqrt{\delta_0}}||(s+\delta_0)W(s)||_\infty^\delta ||(\varphi_1)_t||_2^\delta \tag{19}$$

$$||\zeta_2(t)|| \quad \leq \quad \frac{1}{\sqrt{\delta_0}}||(s+\delta_0)W(s)||_\infty^\delta ||(\varphi_2 u)_t||_2^\delta \tag{20}$$

for $\delta_0 \geq \delta$. In other words

$$||\zeta_1(t)|| \quad \leq \quad K||(\varphi_1)_t||_2^\delta \tag{21}$$

$$||\zeta_2(t)|| \quad \leq \quad K||(\varphi_2)_t||_2^\delta \cdot ||u_t||_2^\delta \tag{22}$$

Assumption 4 *The Lie-derivatives $L_f^\gamma h, ..., L_f h, L_g L_f^{\gamma-1} h$ and the functions $\varphi_1, \varphi_2, f, g,$ and h are smooth and Lipschitz on X. In addition, g and φ_2 are uniformly bounded, i.e. $|g(x)|, |\varphi_2(x)| \leq K$ for all $x \in X$.*

Remark. A Lipschitz function (say $F : X \to R^n$) is bounded by an affine function: $||F(x)|| \leq ||F(0)|| + L||x|| \leq K(1 + ||x||)$ (assuming without loss of generality that $0 \in X$). Moreover, if in addition F is smooth, then $||\partial F/\partial x(x)|| \leq L$ for all $x \in X$.

□

This assumption may restrict the growth of the non-linearities in the system, and it will be given special attention in the discussion section. Under this assumption, we get

$$\begin{aligned} ||(\varphi_1)_t||_2^\delta &\leq K(1 + ||x_t||_2^\delta) \\ ||(\varphi_2)_t||_2^\delta &\leq K \end{aligned}$$

and from (21) and (22) it follows that

$$||\zeta_1(t)|| \quad \leq \quad K(1 + ||x_t||_2^\delta) \tag{23}$$

$$||\zeta_2(t)|| \quad \leq \quad K||u_t||_2^\delta \tag{24}$$

It follows directly from (18) that $||u_t||_2^\delta,\ ||x_t||_2^\delta \leq m_s(t)$. Hence, (23) and (24) proves that

$$||\zeta(t)|| \ \leq \ Km_s(t) \ \leq \ Km(t) \tag{25}$$

Using a similar argument and the Lipschitz-assumptions on the Lie-derivatives, we can show that $|\nu(t)| \leq \overline{\nu}m(t)$ for some constant $\overline{\nu} > 0$, and consequently

$$|\eta(t)| \ \leq \ K\overline{\nu}m(t) \tag{26}$$

Hence, (25) and (26) proves that the normalizing signal has the desired properties (13).

5 Stable Adaptive Control

Before we prove stability of the adaptive control system, we need to make some additional assumptions. Let the vector fields $\tilde{f}, \tilde{g} : X \to R^n$ be defined by

$$\begin{aligned}
\tilde{f}(x) &= f(x) - \frac{L_f^\gamma h(x)}{L_g L_f^{\gamma-1} h(x)} g(x) \\
\tilde{g}(x) &= \frac{1}{L_g L_f^{\gamma-1} h(x)} g(x)
\end{aligned}$$

which are well defined since the system has uniform strong relative degree γ in X.

Assumption 5 *The vector fields $\tilde{g}$, $ad_{\tilde{f}}\tilde{g}, ..., ad_{\tilde{f}}^{\gamma-1}\tilde{g}$ are complete.*

Remark. Completeness of the vector field $\mathrm{ad}_{\tilde{f}}^k\tilde{g}$ holds if the differential equation

$$\dot{x} \ = \ \mathrm{ad}_{\tilde{f}}^k \tilde{g}(x)$$

has a unique solution on $t \in (-\infty, \infty)$, see [24]. For example, if $\mathrm{ad}_{\tilde{f}}^k\tilde{g}$ is globally Lipschitz, the solution to this differential equation cannot have finite escape-time, and the vector field is complete.

□

If the system has uniform strong relative degree $\gamma < n$ on X and Assumption 5 holds, then there exists a diffeomorphism $T : X \to T(X)$ that transforms the state into globally defined canonical coordinates [3] $\xi = (\xi_1^T,\ \xi_2^T)^T \in R^\gamma \times R^{n-\gamma} = R^n$, given by

$$\xi_1 = \begin{pmatrix} \xi_{1,1} \\ \xi_{1,2} \\ \vdots \\ \xi_{1,\gamma} \end{pmatrix} = \begin{pmatrix} y \\ \dot{y} \\ \vdots \\ y^{(\gamma-1)} \end{pmatrix}, \qquad \xi_2 = \begin{pmatrix} \xi_{2,\gamma+1} \\ \vdots \\ \xi_{2,n} \end{pmatrix}$$

where the linearizable dynamics are

$$\dot{\xi}_1 = \begin{pmatrix} \xi_{1,2} \\ \xi_{1,3} \\ \vdots \\ \xi_{1,\gamma-1} \\ L_f^\gamma h(x) + L_g L_f^\gamma h(x) u \end{pmatrix} \tag{27}$$

and the internal dynamics are

$$\dot{\xi}_2 = \psi(\xi_1, \xi_2) \tag{28}$$

In the case when $\gamma = n$, there are no internal dynamics. In general, the feedback linearizing controller can make the internal dynamics unobservable, and instability in the internal dynamics cannot be detected through y. In other words, the system may have a stable input-output behavior, but may be internally unstable. To avoid this problem, we need to assume some kind of stability of the internal dynamics, or more precisely:

Assumption 6 *The unforced zero-dynamics*

$$\dot{\xi}_2 = \psi(0, \xi_2) \tag{29}$$

are exponentially stable, the coordinate transform $T : X \to T(X)$ is Lipschitz on X, and the function $\psi : T(X) \to R^n$ and the inverse coordinate transform $T^{-1} : T(X) \to X$ are both Lipschitz on the set $T(X)$.

Remark. Exponential stability of the zero-dynamics is commonly referred to as a minimum phase property of the system [2]. The term minimum phase is adopted from linear systems theory because of the close relationship between the zero-dynamics and the zeros of a linear system.

□

The final assumption we need to make is

Assumption 7 *The unstructured uncertainty ν, reference trajectory y_M, and initial conditions $x(0)$ and $\hat{\theta}(0)$ are such that $x(t) \in X$ for all $t \geq 0$.*

Remark. If $X = R^n$, then this assumption is always satisfied. If X is a subset of R^n, then transients may in general take $x(t)$ out of X, even if $x(0) \in X$ and there is no unstructured uncertainty. We will discuss this assumption in some detail later.

□

Theorem 2 *Consider the adaptive control system given by the system (2)-(3), the controller (6)-(7) and the adaptive law (15) with normalizing signal (17), and either relative dead-zone, σ-modification or ϵ-modification. If assumptions 1- 7 are satisfied, the design constant c given by (16) and the bound $\overline{\nu}$ on the unstructured uncertainty are both sufficiently small, then $y, x, u, \hat{\theta} \in \mathcal{L}_\infty$. Moreover, there exist constants $\overline{k}_1, \overline{k}_2$, and $\overline{k}_3$ such that for all $t > 0$, the average squared tracking error satisfies*

$$\frac{1}{t}\int_0^t e^2(\tau)d\tau \;\leq\; \overline{k}_1\frac{1}{t} + \overline{k}_2\frac{1}{t}\int_0^t \left(c + \frac{\eta^2(\tau)}{m^2(\tau)}\right) d\tau \;\leq\; \overline{k}_1\frac{1}{t} + \overline{k}_2(c + \overline{k}_3\overline{\nu}^2)$$

Remark. The expression $\overline{k}_2(c + \overline{k}_3\overline{\nu})$ is the asymptotic bound on the average squared tracking error, while $\overline{k}_1/t$ is a bound on the initial transient. In general, $e(t)$ does not go to zero, even in average, but will be small in the average square sense. Notice that some exponentially decaying terms have been neglected in the analysis, since they do not affect the stability of the adaptive controller, and they can in principle be incorporated into the transient bound $\overline{k}_1/t$.

Proof: Given in Appendix B.

□

6 Discussion

Assumptions 1-7 are similar to those needed in the ideal analysis [28, 3], as would be expected. As stated, they are only sufficient, but it has been demonstrated that in some cases, many of them are necessary [29]. It is easy to see that all linear systems, for which all zeros have strictly negative real parts, satisfy all these assumptions for $X = R^n$. The following example illustrates a non-linear system where the Lipschitz assumptions and Assumption 7 are satisfied.

Example

Consider the following system

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -ax_2|x_2| + F(x_3) \\
\dot{x}_3 &= u \\
y &= x_1
\end{aligned}$$

where $F(x_3) = (b\pi/2)\arctan(x_3)$ and $b \geq 0$. If x_1 is interpreted as position, and x_2 as speed, then this is a very simple model of a mechanical system, where $a > 0$ is an aerodynamic or hydrodynamic friction factor, and F represents a power source with saturation. The system has relative degree $\gamma = 3$, there are no internal dynamics, and we easily find

$$\begin{aligned} L_f h(x) &= x_2 \\ L_f^2 h(x) &= -a x_2 |x_2| + F(x_3) \\ L_f^3 h(x) &= 2a^2 x_2^3 - 2a|x_2| F(x_3) \\ L_g L_f^2 h(x) &= \frac{d}{dx_3} F(x_3) = \left(\frac{b\pi}{2}\right) \frac{1}{1+x_3^2} \end{aligned}$$

Obviously, Assumption 4 is not satisfied on $X = R^3$, since x_2^3 and $x_2|x_2|$ are not Lipschitz on R. Consider the Lyapunov-like function

$$V = \frac{1}{2} x_2^2$$

Differentiating along the system trajectory gives

$$\begin{aligned} \dot{V} &= -a x_2^2 |x_2| + (b\pi/2) x_2 \arctan(x_3) \\ &\leq -a x_2^2 |x_2| + b|x_2| \end{aligned}$$

We see that $\dot{V} \leq 0$ for $x_2^2 \geq b/a$. Hence, we conclude $|x_2| \leq \sqrt{b/a}$ (which is reasonable since the speed of a mechanical system with power saturation must be bounded). With $X = R \times \left(-\sqrt{b/a}, \sqrt{b/a}\right) \times R$, we see that Assumption 4 is satisfied. Notice that it is the physics of the system (actuator saturation) that keeps the state x_2 bounded. This means that $x \in X$, independently of $\hat{\theta}$ and u, which means that Assumption 7 is also satisfied as long as $x(0) \in X$. Also notice that X can be redesigned such that bounded disturbances on the force balance can be allowed, since they cannot make x_2 unbounded.

□

The subset X of the state-space in which the assumptions on the nonlinearities hold, is of major importance. We will in the following discuss these assumptions for various structures of X:

- If $X = R^n$, Assumption 7 automatically holds. However, we believe that the class of non-linear systems satisfying the other assumptions on R^n may be somewhat limited. In particular, the Lipschitz assumptions will essentially exclude all systems with states that grow or decay faster than exponentially.
- There exists an interesting class of systems which does not satisfy Assumptions 4, 5 and 6 globally on R^n, but satisfies them on some bounded or unbounded subset X. As the above example illustrates, this does not necessarily imply that the global stability property is lost. This will typically be the case if X is sufficiently large, i.e. it covers the full range of operation of the system.
- If X does not cover all the possible operating conditions of the system, then transients may take $x(t)$ out of X. In order to avoid this, $\hat{\theta}(0)$ should be sufficiently close to θ, the unstructured uncertainty should be sufficiently small, and $x(0)$ must be sufficiently far away from the boundary of X. This may result in a local stability result. Another possibility is to enforce $x(t) \in X$ by feedback control. One way to improve the transient performance and ensure $x(t) \in X$ by feedback is to add a high-gain (sliding) control term with the purpose of enforcing $x(t) \in X$, see [25, 30, 33]. When x is about to leave the set X, a high-gain controller takes over in order to keep x in X. In the SISO case we discuss here, this may sometimes be of limited help, since we are in general only able to control x to a $(n-1)$-dimensional sub-manifold of R^n. In other words, only one state can in general be kept bounded by the high-gain feedback, since there is only one input. In special cases, i.e. when the system is partially in the phase-variable canonical form, a larger number of states can be kept bounded using only one input.

Notice that Assumption 1 may also limit X, since it may be unrealistic to assume the system to have uniform strong relative degree in R^n.

The zero-dynamics are an inherent property of both the system and the choice of output function h. It is well known that there exists an output function h such that $\gamma = n$ (no internal dynamics) in a neighborhood of $x_0 \in X$ if and only if [14]

$$\operatorname{rank}\left[g(x_0),\ \mathrm{ad}_f g(x_0),\ \cdots,\ \mathrm{ad}_f^{n-2} g(x_0),\ \mathrm{ad}_f^{n-1} g(x_0)\right] \quad = \quad n$$

and the distribution

$$\Delta \quad = \quad \operatorname{span}\left\{g,\ \mathrm{ad}_f g,\ \ldots,\ \mathrm{ad}_f^{n-2} g,\ \mathrm{ad}_f^{n-1} g\right\}$$

is involutive near x_0. Even if these rather strong local conditions are satisfied, finding such a h requires the solution of a set of partial differential equations, and the resulting output function h may give an output that is "non-physical" or otherwise unsuited for specifying the control objective. We must therefore expect to have some internal dynamics in many cases.

The various Lipschitz and boundedness assumptions are required for several reasons, including existence and uniqueness of solutions of the differential equations, and the use of input/output stability theory in the proofs.

In the presentation of the results, some of the finite bounds that appear are represented by a generic constant K, in order to simplify the expressions. If explicit constants were used instead, one could in principle calculate a stability margin, i.e. the bound $\overline{\nu}$ on the allowed unstructured uncertainty. One could also explicitly compute the constants in the expression for the asymptotic and transient performance, but this is a considerable problem in practise, and requires significant a priori information about the system. The results should therefore be interpreted qualitatively as a proof that a stability margin and worst-case bounds on the average control performance exist.

7 Concluding Remarks

Here we have addressed the problem of robust adaptive feedback linearizing control of continuous-time minimum phase non-linear systems that are affine in the control input. Under certain assumptions, we have proved robust stability of the adaptive controller. The robustness is with respect to small unstructured unmodeled dynamics and disturbances, and is achieved by normalization and a suitable modification of the adaptive algorithm. If the above mentioned assumptions hold globally, the stability result will also be global. On the other hand, if the assumptions hold on some (possibly unbounded) subset of the state-space, we may still have global or at least local stability. This depends on whether some states of the system are naturally bounded a priori.

The main significance of these results is that the use of weighted $\mathcal{L}_2$-norms and other input/output stability analysis tools that are commonly used for stability and robustness analysis of adaptive control algorithms based on linear models are useful also in the non-linear case.

Appendix A

Lemma 1 *Let $y = H(s)[u]$, where $H(s)$ is a proper transfer function matrix. If $H(s)$ has no poles in the part of the complex plane where $Re(s) \geq -\delta/2$ for some real $\delta \geq 0$, and $u \in \mathcal{L}_{2e}$, then*

$$||y_t||_2^\delta \quad \leq \quad ||H(s)||_\infty^\delta ||u_t||_2^\delta$$

for all $t \geq 0$. If, in addition, $H(s)$ is strictly proper, then

$$||y(t)|| \quad \leq \quad \frac{1}{\sqrt{\delta_0}}||(s+\delta_0)H(s)||_\infty^\delta ||u_t||_2^\delta$$

where $\delta_0 \geq \delta$ is an arbitrary constant.

Proof: The proof is given in e.g. [9]. □

Lemma 2 *(Swapping Lemma). Let $\theta, \varphi : [0,\infty) \to R^n$ and let θ be differentiable. Let $H(s)$ be a proper stable rational transfer function of order n with a minimal state-space realization (A, b, c, d)*

$$H(s) \quad = \quad c(sI - A)^{-1}b + d$$

Then

$$H(s)\left[\theta^T \varphi\right] \quad = \quad \theta^T H(s)[\varphi] - c(sI - A)^{-1}\left[\left((sI - A)^{-1} b \left[\varphi^T\right]\right)\dot{\theta}\right]$$

Proof: A proof may be found in e.g. [9]. □

Lemma 3 *Let $\theta, \varphi : [0,\infty) \to R^n$ be differentiable, and $H(s)$ be a proper stable rational transfer function with relative degree r, stable poles and zeros, and a minimal state-space realization (A, b, c, d). Then*

$$\theta^T\varphi \quad = \quad W_1(s,\alpha)\left[\dot{\theta}^T\varphi + \theta^T\dot{\varphi}\right] + W_2(s)H^{-1}(s)\left[\theta^T H(s)[\varphi] - c(sI - A)^{-1}\left[\left((sI - A)^{-1} b\left[\varphi^T\right]\right)\dot{\theta}\right]\right]$$

where

$$sW_1(s,\alpha) = 1 - W_2(s,\alpha), \qquad W_2(s,\alpha) = \frac{\alpha^k}{(s+\alpha)^k}$$

and $k \geq r$ for an arbitrary constant $\alpha > 0$. Furthermore, for large α we have

$$||W_1(s,\alpha)||_\infty^\delta \quad \leq \quad \frac{K}{\alpha} \tag{30}$$

for any δ that satisfy $0 \leq \delta \leq \delta_0 \ll \alpha$.

Proof: The proof is given in [9, 32]. □

Lemma 4 *Let $y = H(s)[u]$, where $H(s)$ is a stable, strictly proper rational transfer function. If $u \in S(K)$, for some constant K, then $y \in \mathcal{L}_\infty$.*

Proof: The proof is given in [9]. □

Appendix B

Proof of Theorem 2: The proof is structured as follows. In Part I we derive an inequality for the normalizing signal m. Then, in Part II of the proof, we reformulate the inequality from Part I, using the properties of the parameter estimator, into a form that allow us to apply the Bellman-Gronwall lemma to argue that m is a bounded signal if the unstructured uncertainty and the design constant c are not too large. Finally, in Part III we use the swapping lemma in order to establish the bound on the average squared tracking error.

Part I. Here we will prove that

$$m^2(t) \quad \leq \quad K\left(1 + \left(||(\varphi^T\tilde{\theta})_t||_2^\delta\right)^2 + \left(||\nu_t||_2^\delta\right)^2\right) \tag{31}$$

It is immediately clear from (18) that

$$m(t) \quad \leq \quad K\left(1 + ||u_t||_2^\delta + ||x_t||_2^\delta\right) \tag{32}$$

To arrive at (31), we need to find bounds on $||u_t||_2^\delta$ and $||x_t||_2^\delta$. The control signal satisfies

$$\varphi_{g,f}^T\hat{\theta}_{g,f}u \quad = \quad M^{-1}(s)\left[y_M\right] - \left(\varphi_{f,\gamma}^T\hat{\theta}_{f,\gamma} + \alpha_{\gamma-1}\varphi_{f,\gamma-1}^T\hat{\theta}_{f,\gamma-1} + ... + \alpha_1\varphi_{f,1}^T\hat{\theta}_{f,1} + \alpha_0 y\right)$$

or

$$\varphi_2^T\hat{\theta}_2 u \quad = \quad M^{-1}(s)\left[y_M\right] - \alpha_0 y - \varphi_1^T\hat{\theta}_1$$

By assumption 2, $M^{-1}(s)\,[y_M]$ is a bounded signal. The behavior of the closed loop system is in addition to the adaptive law and internal dynamics described by

$$\begin{aligned} u &= \frac{M^{-1}(s)y_M - \alpha_0 y - \varphi_1^T\hat{\theta}_1}{\varphi_2^T\hat{\theta}_2} && (33)\\ e &= M(s)\left[\varphi^T\tilde{\theta} + \nu\right] && (34) \end{aligned}$$

Differentiating the error equation (34) gives

$$\begin{aligned} y &= y_M - M(s)\left[\varphi^T\tilde{\theta} + \nu\right]\\ \dot{y} &= \dot{y}_M - sM(s)\left[\varphi^T\tilde{\theta} + \nu\right]\\ &\vdots\\ y^{(\gamma-1)} &= y_M^{(\gamma-1)} - s^{\gamma-1}M(s)\left[\varphi^T\tilde{\theta} + \nu\right] \end{aligned}$$

Hence,

$$\xi_1 = \begin{pmatrix} y_M \\ \dot{y}_M \\ \vdots \\ y_M^{(\gamma-1)} \end{pmatrix} - M(s)\begin{pmatrix} 1 \\ s \\ \vdots \\ s^{\gamma-1} \end{pmatrix}\left[\varphi^T\tilde{\theta} + \nu\right] \tag{35}$$

Since $M(s), sM(s), ..., s^{\gamma-1}M(s)$ are all proper, stable transfer functions, and assumption 2 is satisfied, it follows from (35) that

$$||(\xi_1)_t||_2^\delta \;\leq\; K\left(1 + ||(\varphi^T\tilde{\theta})_t||_2^\delta + ||\nu_t||_2^\delta\right)$$

Following [28], we argue that since the zero-dynamics are exponentially stable, by Lyapunovs converse theorem [18], there exists a differentiable function $V : R^{n-\gamma} \to [0,\infty)$ that satisfies

$$\begin{aligned} k_1||\xi_2||^2 &\leq V(\xi_2) \leq k_2||\xi_2||^2 && (36)\\ \frac{dV}{d\xi_2}\cdot\psi(0,\xi_2) &\leq -k_3||\xi_2||^2\\ \left|\left|\frac{dV}{d\xi_2}\right|\right| &\leq k_4||\xi_2|| \end{aligned}$$

for some strictly positive constants $k_1, ..., k_4$. This gives

$$\begin{aligned} \dot{V} &= \frac{dV}{d\xi_2}\cdot\psi(\xi_1,\xi_2)\\ &\leq -k_3||\xi_2||^2 + \frac{dV}{d\xi_2}(\psi(\xi_1,\xi_2) - \psi(0,\xi_2))\\ &\leq -k_3||\xi_2||^2 + k_4L_1||\xi_2||\cdot||\xi_1|| \end{aligned}$$

where L_1 is a Lipschitz-constant for ψ. We conclude that $\dot{V} \leq 0$ for $||\xi_2|| \geq (k_4L_1/k_3)||\xi_1||$. Hence, by (36) it follows that $||\xi_2|| \leq K||\xi_1||$, and

$$||\xi_t||_2^\delta \;\leq\; ||(\xi_1)_t||_2^\delta + ||(\xi_2)_t||_2^\delta \;\leq\; K||(\xi_1)_t||_2^\delta \;\leq\; K\left(1 + ||(\varphi^T\tilde{\theta})_t||_2^\delta + ||\nu_t||_2^\delta\right)$$

Since T^{-1} is Lipschitz, we get $||x|| \leq K(1 + ||\xi||)$ and

$$||x_t||_2^\delta \;\leq\; K(1 + ||\xi_t||_2^\delta) \;\leq\; K\left(1 + ||(\varphi^T\tilde{\theta})_t||_2^\delta + ||\nu_t||_2^\delta\right) \tag{37}$$

Then we get from the equation for the controller (33), using $\hat{\theta} \in \mathcal{L}_\infty$

$$|u| \;\leq\; \beta_0^{-1}\left(K + |\alpha_0|\cdot|y| + K||\varphi_1(x)||\right) \tag{38}$$

Since assumption 4 gives $||y_t||_2^\delta \le K(1+||x_t||_2^\delta)$, we get

$$\begin{aligned} ||u_t||_2^\delta &\le \beta_0^{-1}\left(K + |\alpha_0|\cdot ||y_t||_2^\delta + K||(\varphi_1)_t||_2^\delta\right) \\ &\le K\left(1+||x_t||_2^\delta\right) \end{aligned} \tag{39}$$

Hence, (31) follows from (37) and (39).

Part II. We proceed by proving that $m \in \mathcal{L}_\infty$. Using Lemma 3, we can rewrite

$$\varphi^T\tilde{\theta} = W_1(s,\alpha)\left[\dot{\varphi}^T\tilde{\theta} + \varphi^T\dot{\tilde{\theta}}\right] + W_2(s,\alpha)M^{-1}(s)\left[\zeta^T\tilde{\theta} - c^T(sI-A)^{-1}\left[\pi^T\dot{\tilde{\theta}}\right]\right] \tag{40}$$

where $\pi = (sI-A)^{-1}b\,[\varphi]$ and (A,b,c,d) is a minimal state-space realization of $M(s)$. Moreover, for some yet unspecified real constant $\alpha > 0$ we have defined

$$W_2(s,\alpha) = \frac{\alpha^\gamma}{(s+\alpha)^\gamma}, \qquad sW_1(s,\alpha) = 1 - W_2(s,\alpha)$$

It follows directly from the Lipschitz assumptions and (38) that

$$\begin{aligned} ||\dot{x}|| &\le K(1+|u|+||x||)|| \;\le\; K(1+||x||) \\ ||\varphi|| &\le K(1+||x||) \\ ||\dot{\varphi}|| &\le \left|\left|\frac{\partial\varphi}{\partial x}\right|\right|\cdot||\dot{x}|| \;\le\; K(1+||x||) \end{aligned}$$

We rewrite (2) as

$$\dot{x} = -\frac{\delta}{2}x + \left(g(x)u + f(x) + \frac{\delta}{2}x\right)$$

or

$$x = \left(sI + \frac{\delta}{2}I\right)^{-1}\left[g(x)u + f(x) + \frac{\delta}{2}x\right]$$

Since $(sI+\delta/2I)^{-1}$ is strictly proper, we get from Lemma 1 and assumption 4 that

$$||x(t)|| \le K(1+||u_t||_2^\delta + ||x_t||_2^\delta) \;\le\; K\left(1+||x_t||_2^\delta\right) \;\le\; Km(t) \tag{41}$$

Now, it is clear that

$$\left|\frac{\dot{\varphi}}{m}\right|, \left|\frac{\varphi}{m}\right|, \left|\frac{\zeta}{m}\right|, \left|\frac{\pi}{m}\right| \in \mathcal{L}_\infty$$

Hence, from (30) and (40) we get for sufficiently large α

$$||(\varphi^T\tilde{\theta})_t||_2^\delta \le \frac{K}{\alpha}\left(||(\tilde{\theta}m)_t||_2^\delta + ||(\dot{\tilde{\theta}}m)_t||_2^\delta\right) + K\left(||(\varepsilon m^2)_t||_2^\delta + \left|\left|\left(\frac{\eta}{m}m\right)_t\right|\right|_2^\delta + ||(\dot{\tilde{\theta}}m)_t||_2^\delta\right)$$

and

$$\left(||(\varphi^T\tilde{\theta})_t||_2^\delta\right)^2 \le K\int_0^t e^{-\delta(t-\tau)}m^2(\tau)\left(\frac{1}{\alpha^2} + (\varepsilon(\tau)m(\tau))^2 + \dot{\tilde{\theta}}(\tau)^T\dot{\tilde{\theta}}(\tau) + \frac{\eta^2(\tau)}{m^2(\tau)}\right)d\tau$$

Let us define the signal

$$\kappa^2 = \dot{\tilde{\theta}}^T\dot{\tilde{\theta}} + (\varepsilon m)^2 + \frac{\eta^2}{m^2} + \frac{1}{\alpha^2}$$

and substitute the above expression into (31):

$$e^{\delta t}m^2(t) \le Ke^{\delta t} + K\int_0^t e^{\delta\tau}m^2(\tau)\kappa^2(\tau)d\tau$$

Using the estimator properties $\dot{\tilde{\theta}}, \varepsilon m \in S(c+\eta^2/m^2)$ (or the stronger property $\dot{\tilde{\theta}} \in \mathcal{L}_1 \cap \mathcal{L}_2$ if relative dead-zone is applied), and the facts that $|\eta(t)| \leq K\overline{\nu}m(t)$ and α can be chosen arbitrarily large, we get

$$\int_0^t \kappa^2(\tau)d\tau \quad \leq \quad K + K\int_0^t (c + K\overline{\nu}^2 + 1/\alpha^2)d\tau \;\leq\; K(1+(c+K\overline{\nu}^2)t)$$

Using the Bellman-Gronwall lemma, e.g. [6], we get

$$e^{\delta t}m^2(t) \quad \leq \quad Ke^{\delta t} + K\int_0^t e^{\delta\sigma}\kappa^2(\sigma)\exp\left(K\int_\sigma^t \kappa^2(\tau)d\tau\right)d\sigma$$

Hence,

$$m^2(t) \leq K + K\int_0^t \exp\left(-(\delta-(c+K\overline{\nu}^2))(t-\sigma)\right)\kappa^2(\sigma)d\sigma$$

Defining $\delta_1 = \delta - (c+K\overline{\nu}^2)$, we get

$$m^2(t) \leq K + K\int_0^t e^{-\delta_1(t-\sigma)}\kappa^2(\sigma)d\sigma \tag{42}$$

Notice that $c>0$ is a design-constant for the estimator, and can can be made arbitrary small if σ- or ϵ-modification is used. With relative dead-zone, we cannot make c smaller than $K\overline{\nu}$. Anyway, since the unmodeled dynamics are sufficiently small, then $\delta_1 > 0$, and it follows from (42) and Lemma 4 that $m \in \mathcal{L}_\infty$, since $\kappa^2 \in S(K)$. The parameter estimator guarantees $\hat{\theta} \in \mathcal{L}_\infty$, and boundedness of m implies $y, x, u \in \mathcal{L}_\infty$, since they in general are bounded by m, cf. (41) and (38).

Part III. The prediction error is

$$\varepsilon_1 = m^2\varepsilon = W(s)[\varphi^T]\tilde{\theta} + W(s)[\nu] \tag{43}$$

and the tracking error is given by

$$e \quad = \quad M(s)\left[\varphi^T\tilde{\theta}\right] + M(s)[\nu]$$

Using the Swapping Lemma (Lemma 2 in the Appendix), this can be rewritten as

$$e = M(s)[\varphi^T]\tilde{\theta} + M(s)[\nu] - c^T(sI-A)^{-1}\left[\left((sI-A)^{-1}b\left[\varphi^T\right]\right)\dot{\tilde{\theta}}\right] \tag{44}$$

where (A,b,c,d) is a minimal state-space realization of $M(s)$. From (43) and (44) we get

$$e \quad = \quad M(s)W^{-1}(s)\left[m^2\varepsilon - c^T(sI-A)^{-1}\left[\left((sI-A)^{-1}b\left[\varphi^T\right]\right)\dot{\tilde{\theta}}\right]\right]$$

Since $M(s)W^{-1}(s)$ is a proper, stable transfer function, and m is bounded, we get

$$||e_t||_2 \quad \leq \quad K||M(s)W^{-1}(s)||_\infty \left(||(\varepsilon m)_t||_2 + ||\dot{\tilde{\theta}}_t||_2\right)$$

or

$$\int_0^t e^2(\tau)d\tau \quad \leq \quad K\int_0^t \kappa^2(\tau)d\tau \;\leq\; K + K\int_0^t\left(c + \frac{\eta^2(\tau)}{m^2(\tau)}\right)d\tau \;\leq\; K + K(c+K\overline{\nu}^2)t$$

□

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