Regularized Nonlinear Moving Horizon Observer for Detectable Systems ⋆

Sui Dan and Tor A. Johansen

Department of Engineering Cybernetics, Norwegian University of Science and Technology, Trondheim, Norway.

Abstract: A moving horizon state observer is developed for nonlinear discrete-time systems. The new algorithm is proved to converge exponentially under a strong detectability assumption and the data being persistently exciting. However, in many practical estimation problems, such as combined state and parameter estimation, data may not be exciting for every period of time. The algorithm therefore has regularization mechanisms to ensure graceful degradation of performance in cases when data are not exciting and data are corrupted by noise. This includes the use of a priori estimates in the moving horizon cost function, and the use of thresholded singular value decomposition to avoid ill-conditioned or ill-posed inversion of the associated nonlinear algebraic equations that forms the basis of the moving horizon state estimate.

Keywords: Moving Horizon Estimation; Nonlinear Systems; Regularization, Filtering.

1. INTRODUCTION

Consider the state estimation problem of nonlinear discrete-time systems. A Moving Horizon State Estimator (MHE) makes use of a finite memory moving window of both current and historical measurement data in the least-squares criterion, possibly in addition to an a priori state estimate and covariance matrix estimate to set the initial cost at the beginning of the data window, see Rao et al. [2003], Moraal and Grizzle [1995a], Alessandri et al. [2008] for different formulation relying on somewhat different assumptions. Such a moving horizon estimator can also be considered a sub-optimal approximation to an estimator that uses the full history of past data, and some empirical studies (Haseltine and Rawlings [2005]) show that the MHE can perform better than the Extended Kalman Filter (EKF) in terms of accuracy and robustness.

Uniform observability (in some form, see also Raff et al. [2005], Alamir [1999], Sontag [1984]) is assumed for stability or convergence proofs in the above mentioned references, including the EKF, see Reif et al. [1998], Reif and Unbehauen [1999]. Uniform observability ensures that the system and data are such that the problem of inverting the nonlinear algebraic equations is well-posed in the sense of (Tikhonov and Arsenin [1977]), i.e. that the state estimate solution exists, is unique and depends continuously on the measurement data. In the context of optimization this is commonly referred to as stability of the solution, that can be guaranteed under certain regularity and rank conditions (Fiacco [1983]). Such robustness is essential in any practical application since otherwise the estimates will be impossible to compute, and may be divergent or highly sensitive to imperfections such as numerical round-off errors, model error, disturbances, quantization and measurement noise. However, uniform observability is a restrictive assumption that is likely not to hold in certain interesting and important state estimation applications. This is in particular true for combined state and parameter estimation problems where convergence of the parameter estimates in an augmented state space model will often depend on the information contents in the data, typically formulated as a condition for persistently exciting (PE) input data appearing in adaptive control and estimation, e.g. Kreisselmeier [1977], Krstic et al. [1995], or boundedness of the EKF covariance matrix estimate (Reif et al. [1998], Reif and Unbehauen [1999]). In many practical applications the data will be sufficiently exciting for periods of time, but may in some time intervals contain insufficient excitation and information. It should also be noted that with some exceptions (e.g. Panteley et al. [2001], Sedoglavic [2002]), both uniform observability and PE conditions are difficult to verify a priori.

In this paper we consider strongly detectable systems (Moraal and Grizzle [1995b]), and the objective and novel contribution of the present work is to provide and study an MHE method based on Alessandri et al. [2008] with modifications that give acceptable practical performance also when the condition of uniform observability is violated due to temporary lack of informative data or the system being strongly detectable rather than uniformly observable. Following the spirit of Moraal and Grizzle [1995b] we introduce practical regularization mechanisms that monitors and estimates the information contents and degree of excitation in the data, and take corresponding action by adaptively weighting the measured data and a priori estimates from the dynamic model. Although the moving horizon state estimation formulation based on Alessandri et al. [2008] does not rely on an explicit uncertainty estimate in terms of a covariance matrix estimate (unlike formulations that may apply an arrival cost estimate, Rao et al. [2003]), the monitoring of persistent excitation in the moving horizon nonlinear observer relies on a related Hessian matrix estimate. This makes the approach similar in spirit to well known practically motivated modifications of the EKF and Recursive Least Squares estimation methods that rely on monitoring and resetting of the covariance matrix estimate, Salgado et al. [1988], directional forgetting, Parkum et al. [1992], and using singular value decomposition

⋆ This work was supported by the Research Council of Norway under the Strategic University Program on Computational Methods in Nonlinear Motion Control.

for numerically robust matrix inversion, Golub and van Loan [1983].

The following notation and nomenclature is used. For a vector $x \in R^n$, let $||x|| = \sqrt{x^T x}$ denote the Euclidean norm, and $||x||_M = \sqrt{x^T M x}$ the weighted Euclidean norm when $M = M^T$ is a positive definite matrix. Recall that the induced matrix norm $||M|| = \overline{\sigma}(M)$ equals the largest singular value of M. For two vectors $x \in R^n$ and $y \in R^m$ we let $\mathrm{col}(x,y)$ denote the column vector in R^{n+m} where x and y are stacked into a single column. The Moore-Penrose pseudo-inverse (Golub and van Loan [1983]) of a matrix M is denoted M^+ and we recall that for a matrix M of full rank it is given by $M^+ = (M^T M)^{-1} M^T$ while in general it is defined as $M^+ = VS^+U^T$ where $M = USV^T$ is a singular value decomposition where S is a diagonal matrix with the singular values $\sigma_1, ..., \sigma_n$ on the diagonal, and S^+ is the diagonal matrix $S = \mathrm{diag}(1/\sigma_1, ..., 1/\sigma_r, 0, ..., 0)$ where $r \le n$ of the singular values are non-zero. The composition of two functions f and g is written $f \circ g(x) = f(g(x))$. Finally, a function $\varphi : R^+ \to R$ is called a K-function if $\varphi(0) = 0$ and it is strictly increasing.

2. NONLINEAR MOVING HORIZON ESTIMATION PROBLEM FORMULATION

Consider the following discrete-time nonlinear system:

$$x_{t+1} = f(x_t, u_t) \tag{1a}$$
$$y_t = h(x_t, u_t), \tag{1b}$$

where $x_t \in \mathbb{X} \subseteq R^{n_x}$, $u_t \in \mathbb{U} \subseteq R^{n_u}$ and $y_t \in R^{n_y}$ are respectively the state, input and measurement vectors, and t is the discrete time index. The $N+1$ consecutive measurements of outputs and inputs until time t are denoted as

$$Y_t = \begin{bmatrix} y_{t-N} \\ y_{t-N+1} \\ \vdots \\ y_t \end{bmatrix}, \quad U_t = \begin{bmatrix} u_{t-N} \\ u_{t-N+1} \\ \vdots \\ u_t \end{bmatrix}. \tag{2}$$

To express Y_t as a function of x_{t-N} and U_t, denote $f^{u_t}(x_t) = f(x_t, u_t)$ and $h^{u_t}(x_t) = h(x_t, u_t)$, and note from (1b) that the following algebraic map can be formulated (Moraal and Grizzle [1995a]):

$$Y_t = H(x_{t-N}, U_t) = H_t(x_{t-N}) = \begin{bmatrix} h^{u_{t-N}}(x_{t-N}) \\ h^{u_{t-N+1}} \circ f^{u_{t-N}}(x_{t-N}) \\ \vdots \\ h^{u_t} \circ f^{u_{t-1}} \circ \cdots \circ f^{u_{t-N}}(x_{t-N}) \end{bmatrix}. \tag{3}$$

Definition 1. The system (1a)-(1b) is *N-observable* if there exists a K-function φ such that for all $x_1, x_2 \in \mathbb{X}$ there exists a feasible $U_t \in \mathbb{U}^{N+1}$ such that

$$\varphi(||x_1 - x_2||^2) \le ||H(x_1, U_t) - H(x_2, U_t)||^2.$$

Definition 2a. The input $U_t \in \mathbb{U}^{N+1}$ is said to be *N-exciting* for the N-observable system (1a)-(1b) at time t if there exists a K-function φ_t such that for all $x_1, x_2 \in \mathbb{X}$ satisfying

$$\varphi_t(||x_1 - x_2||^2) \le ||H(x_1, U_t) - H(x_2, U_t)||^2.$$

The following regularity properties are assumed throughout this paper:

(A1) $\{y_0, y_1, y_2, ...\}$ and $\{u_0, u_1, u_2, ...\}$ are bounded sequences.

(A2) The functions f and h are twice differentiable on the closed set $\mathbb{X} \times \mathbb{U}$.

(A3) $\mathbb{X}$ and $\mathbb{U}$ are compact and convex sets.

(A4) The initial state x_0 and the control sequence $\{u_t\}$ are such that the system trajectory $\{x_t\}$ lies in $\mathbb{X}$.

Like in the linear case, an observability rank condition can be formulated (see also Moraal and Grizzle [1995a], Alessandri et al. [2008], Fiacco [1983] for similar results):

Lemma 1. If (A2)-(A3) are satisfied and the Jacobian matrix $\frac{\partial H}{\partial x}(x, U_t)$ has full rank (equal to n_x) for all $x \in \mathbb{X}$ and some $U_t \in \mathbb{U}^{N+1}$ then the system is N-observable and the input U_t is N-exciting for the system (1a)-(1b) at time t.

Proof. From Proposition 2.4.7 in Abraham et al. [1983], we have

$$H(x_1, U_t) - H(x_2, U_t) = \Phi_t(x_1, x_2)(x_1 - x_2), \tag{4}$$

where

$$\Phi_t(x_1, x_2) = \int_0^1 \frac{\partial}{\partial x} H((1-s)x_2 + s x_1, U_t) ds. \tag{5}$$

Due to the observability rank condition being satisfied, $\Phi_t^T(\cdot)\Phi_t(\cdot) > 0$ and the system of nonlinear algebraic equation (4) can be inverted

$$x_1 - x_2 = \Phi_t^+(x_1, x_2)\big(H(x_1, U_t) - H(x_2, U_t)\big),$$

where $\Phi_t^+(x_1, x_2) = (\Phi_t^T(x_1, x_2)\Phi_t(x_1, x_2))^{-1}\Phi_t^T(x_1, x_2)$, and

$$\frac{1}{\pi_t^2(x_1, x_2)} \|x_1 - x_2\|^2 \le \|H(x_1, U_t) - H(x_2, U_t)\|^2,$$

where $\pi_t(x_1, x_2) = \|\Phi_t^+(x_1, x_2)\|$. This proves that the conditions in Definitions 1 and 2a hold with $\varphi(s) = s/\overline{p}^2$ where $\overline{p} = \sup_{x_1, x_2 \in \mathbb{X}, U \in \mathbb{U}} ||\Phi^+(x_1, x_2, U)||$ is bounded due to (A2) and (A3). $\square$

Define the N-information vector at time t as

$$I_t = col(y_{t-N}, \ldots, y_t, u_{t-N}, \ldots, u_t). \tag{6}$$

When a system is not N-observable, it is not possible to reconstruct exactly all the state components from the N-information vector. However, in some cases one may be able to reconstruct exactly at least some components, based on the N-information vector, and the remaining components can be reconstructed asymptotically. This corresponds to the notion of detectability (Moraal and Grizzle [1995b]), where we suppose there exists a coordinate transform $\mathbb{T} : \mathbb{X} \to \mathbb{D} \subseteq R^{n_x}$

$$d = \begin{pmatrix} \xi \\ z \end{pmatrix} = \mathbb{T}(x) \tag{7}$$

such that (1a)-(1b) is equivalent to

$$\xi_{t+1} = F_1(\xi_t, z_t, u_t) \tag{8a}$$
$$z_{t+1} = F_2(z_t, u_t) \tag{8b}$$
$$y_t = G(z_t, u_t). \tag{8c}$$

This transform effectively partitions the state x into an observable state z and an unobservable state ξ, and the following strong detectability definition can be given, Moraal and Grizzle [1995b]:

Definition 3. The system (1a)-(1b) is *strongly N-detectable* if

(1) There exists a coordinate transform $\mathbb{T} : \mathbb{X} \to \mathbb{D}$ that brings the system in the form (8a)-(8c).
(2) The sub-system (8b)-(8c) is N-observable.
(3) The sub-system (8a) has uniformly contractive dynamics, i.e. there exists a constant $L_1 < 1$ such that for all $[\xi_1, z] \in \mathbb{D}, [\xi_2, z] \in \mathbb{D}$ and $u \in \mathbb{U}$ the function F_1 satisfies

$$||F_1(\xi_1, z, u) - F_1(\xi_2, z, u)||_P \le L_1 ||\xi_1 - \xi_2||_P. \tag{9}$$

for some symmetric matrix $P > 0$.

Definition 2b. The input U_t is said to be *N-exciting* for a strongly N-detectable system (1a)-(1b) at time t if it is N-exciting for the associated sub-system (8b)-(8c) at time t.

N-excitation on the input U_t may be difficult to assess a priori. In this paper we will study how a level of N-excitation can be monitored online, and used in modifications to the basic MHE when this requirement is violated because the input data are not N-exciting at all times.

The proposed MHE problem consists in estimating, at any time $t = N, N+1, \ldots$, the state vectors $x_{t-N}, \ldots, x_t$, on the basis of a priori estimates $\overline{x}_{t-N,t}, \ldots, \overline{x}_{t,t}$ and the information vector I_t. It is assumed that an a priori estimator is determined from the last estimate $\hat{x}^o_{t-N-1,t-1}$ via the application of the function f, that is,

$$\overline{x}_{i+1,t} = f(\overline{x}_{i,t}, u_i), \;\; i = t-N, \ldots, t-1, \tag{10}$$

where $\overline{x}_{t-N,t} = f(\hat{x}^o_{t-N-1,t-1}, u_{t-N-1})$. A convergent estimator is pursued by the following weighted regularized least-squares criterion

$$J(\hat{x}_{t-N,t}, \overline{x}_{t-N,t}, I_t) = \|W_t(Y_t - H_t(\hat{x}_{t-N,t}))\|^2 + \sum_{i=t-N}^{i=t} \beta_{i-t+N} \|\hat{x}_{i,t} - \overline{x}_{i,t}\|^2 \tag{11a}$$

$$s.t. \;\; \hat{x}_{i+1,t} = f(\hat{x}_{i,t}, u_i), \;\; i = t-N, \ldots, t-1 \tag{11b}$$

with $\beta_0 > 0$, $\beta_i \geq 0, i = 1, \ldots, N$ and W_t being a time-varying weight matrix. Let $J^o_t = \min_{\hat{x}_{t-N,t}} J(\hat{x}_{t-N,t}, \overline{x}_{t-N,t}, I_t)$ subject to (11b), let $\hat{x}^o_{t-N,t}$ be the associated optimal estimate, and the estimation error is defined as

$$e_{t-N} = x_{t-N} - \hat{x}^o_{t-N,t}. \tag{12}$$

This formulation is a slight variation of Alessandri et al. [2008] since the deviation between the state estimate and a priori estimate is weighted on the whole horizon, rather than just at the beginning of the horizon. Furthermore, an adaptive method for choosing W_t based on monitoring of excitation is introduced in section 4.

3. CONVERGENCE OF REGULARIZED NONLINEAR MOVING HORIZON ESTIMATOR

(A2b) The functions F_1, F_2 and g are twice differentiable on $\overline{\mathbb{D}} \times \mathbb{U}$, where $\overline{\mathbb{D}}$ is the closed convex hull of the set $\mathbb{D} = \mathbb{T}(\mathbb{X})$.

(A5) The system (1a)-(1b) is strongly N-detectable and the input U_t is N-exciting for all time $t \geq 0$.

In order to state the convergence, the following definitions are given:

$$L_2 = \max_{d \in \overline{\mathbb{D}}, u \in \mathbb{U}} || \frac{\partial F_1}{\partial z}(\xi, z, u)||, \; L_3 = \max_{d \in \overline{\mathbb{D}}, u \in \mathbb{U}} || \frac{\partial F_2}{\partial z}(z, u)||,$$

$$k_{T^{-1}} = \max_{d \in \overline{\mathbb{D}}} || \frac{\partial \mathbb{T}^{-1}}{\partial d}(d)||, \; k_T = \max_{x \in \mathbb{X}} || \frac{\partial \mathbb{T}}{\partial x}(x)||,$$

$$c_h = \sum_{i=0}^{N} \beta_i k^2_{T^{-1}} a_i + \sum_{i=0}^{N} \beta_i k^2_{T^{-1}} L_3^{2(i+1)}$$

with $a_{i+1} = L_1^2 a_i + L_2^2 L_3^{2(i+1)}, a_0 = 0,$

$$G(z_{t-N}, U_t) = G_t(z_{t-N})$$
$$= \mathrm{col}(g(z_{t-N}), \cdots, g \circ F_2^{u_{t-1}} \circ \cdots \circ F_2^{u_{t-N}}(z_{t-N})),$$
$$\Phi^o_t(z_{t-N}, \hat{z}^o_{t-N,t}) = \int_0^1 \frac{\partial}{\partial z} G((1-s)z_{t-N} + s\hat{z}^o_{t-N,t}, U_t) ds,$$
$$p_{z,t} = p_t(z_{t-N}, \hat{z}^o_{t-N,t}) = \|(W_t \Phi^o_t(z_{t-N}, \hat{z}^o_{t-N,t}))^+\|,$$
$$q_{z,t} = c_h p^2_{z,t}.$$

In the computation of the above bounds, it is implicitly assumed that $\mathbb{X}$ is chosen large enough such that $\hat{x}^o_{t-N,t} \in \mathbb{X}$. Alternatively, constraints could be implemented in the optimization to enforce this condition to hold.

Theorem 1. Suppose that assumptions (A1)-(A5) hold. Then there exists a sufficiently large weight matrix W_t such that $q_{z,t} < 1$ and the error $e_t \to 0$ exponentially when $t \to \infty$.

Proof. The basic idea behind the proof consists in looking for upper and lower bounds on the optimal cost J^o_t.

Lower bound on the optimal cost J^o_t

Using the fact that the system (1) can be transformed using (7), there exist $d_{t-N} = \mathbb{T}(x_{t-N})$, $\hat{d}^o_{t-N} = \mathbb{T}(\hat{x}^o_{t-N})$ and $\overline{d}_{t-N,t} = \mathbb{T}(\overline{x}_{t-N,t})$ such that in the new coordinates, the system is in the form of (8a)-(8c). Note that the least squares term in the right-hand side of expression (11a) in the new coordinations can be rewritten as

$$\|W_t(Y_t - G_t(\hat{z}^o_{t-N,t}))\|^2 = \|W_t(G_t(z_{t-N}) - G_t(\hat{z}^o_{t-N,t}))\|^2.$$

From arguments similar to Lemma 1, it is clear that W_t can be chosen such that $p^2_{z,t}$ is uniformly bounded by any chosen positive number, and

$$\|W_t(Y_t - G(\hat{z}^o_{t-N,t}, U_t))\|^2 \geq 1/p^2_{z,t} \|z_{t-N} - \hat{z}^o_{t-N,t}\|^2. \tag{13}$$

Taking zero as the lower bound on the second term of (11a) we get $J^o_t \geq 1/p^2_{z,t} \|z_{t-N} - \hat{z}^o_{t-N,t}\|^2$.

Upper bound on the optimal cost J^o_t

Let $\check{x}_{t-N} = \mathbb{T}^{-1}(\check{d}_{t-N})$, $\check{d}_{t-N} = [\check{\xi}_{t-N}, \check{z}_{t-N}]^T = [\overline{\xi}_{t-N,t}, z_{t-N}]^T$. From the optimality of $\hat{x}^o_{t-N,t}$, we have $J^o_t \leq J(\check{x}_{t-N}, \overline{x}_{t-N,t}, I_t)$. It is easy to see that $\|W_t(Y_t - H(\check{x}_{t-N}, U_t))\|^2 = \|W_t(Y_t - G(z_{t-N}, U_t))\|^2 = 0$. And

$$||\check{x}_i - \overline{x}_{i,t}||^2 \leq k^2_{T^{-1}} ||\check{d}_i - \overline{d}_{i,t}||^2 = k^2_{T^{-1}} (||\check{\xi}_i - \overline{\xi}_{i,t}||^2 + ||z_i - \overline{z}_{i,t}||^2),$$

we have $J(\check{x}_{t-N}, \overline{x}_{t-N,t}, I_t) \leq \sum_{i=t-N}^{i=t} \beta_{i-t+N} k^2_{T^{-1}} (||\check{\xi}_i - \overline{\xi}_{i,t}||^2 + ||z_i - \overline{z}_{i,t}||^2)$. When $i = t-N$, since $\check{\xi}_i = \overline{\xi}_{i,t}$,

$$||\check{\xi}_i - \overline{\xi}_{i,t}||^2 + ||z_i - \overline{z}_{i,t}||^2 \leq L_3^2 \|z_{t-N-1} - \hat{z}^o_{t-N-1,t-1}\|^2.$$

When $i = t-N+1$, it is easy to show that

$$||\check{\xi}_i - \overline{\xi}_{i,t}||^2 = ||F_1(\check{\xi}_{i-1}, z_{i-1}, u_{i-1}) - F_1(\overline{\xi}_{i-1,t}, \overline{z}_{i-1,t}, u_{i-1})||^2$$
$$= \|F_1(\check{\xi}_{i-1}, z_{i-1}, u_{i-1}) - F_1(\overline{\xi}_{i-1,t}, z_{i-1}, u_{i-1})$$
$$+ F_1(\overline{\xi}_{i-1,t}, z_{i-1}, u_{i-1}) - F_1(\overline{\xi}_{i-1,t}, \overline{z}_{i-1,t}, u_{i-1})\|^2$$
$$\leq L_2^2 ||z_{t-N} - \overline{z}_{t-N,t}||^2 \leq L_2^2 L_3^2 \|z_{t-N-1} - \hat{z}^o_{t-N-1,t-1}\|^2.$$

Then we have $||\check{\xi}_i - \overline{\xi}_{i,t}||^2 + ||z_i - \overline{z}_{i,t}||^2 \leq (L_2^2 L_3^2 + L_3^4) \|z_{t-N-1} - \hat{z}^o_{t-N-1,t-1}\|^2$. From the above derivation, it is easy to obtain that

$$J(\check{x}_{t-N}, \overline{x}_{t-N,t}, I_t) \leq c_h \|z_{t-N-1} - \hat{z}^o_{t-N-1,t-1}\|^2.$$

Proof of the convergence

Consequently, there always exists a sufficiently large W_t such that $q_{z,t} = c_h p^2_{z,t} < 1$ and

$$\|z_{t-N} - \hat{z}^o_{t-N,t}\|^2 \leq q_{z,t} \|z_{t-N-1} - \hat{z}^o_{t-N-1,t-1}\|^2. \tag{14}$$

Considering the MHE problem (11), we know that $\hat{\xi}^o_{t-N,t} = \overline{\xi}_{t-N,t}$. From Definition 3

$$||\xi_{t-N}-\hat{\xi}^o_{t-N,t}\|^2_P \le L_1^2\|\xi_{t-N-1}-\hat{\xi}^o_{t-N-1,t-1}\|^2_P + L_2^2\|z_{t-N-1}-\hat{z}^o_{t-N-1,t-1}\|^2_P.$$

Then when $i=t-N\ge k$

$$||\xi_i-\hat{\xi}^o_{i,t}\|^2_P \le \mu^k\|\xi_{i-k}-\hat{\xi}^o_{i-k,t-k}\|^2_P + kL_2^2\mu^{k-1}\|z_{i-k}-\hat{z}^o_{i-k,t-k}\|^2_P$$
$$\|z_i-\hat{z}^o_{i,t}\|^2 \le \mu^k\|z_{i-k}-\hat{z}^o_{i-k,t-k}\|^2,$$

where $q_z=\sup_{t\ge 0} q_{z,t}$, and $\mu=\max\{q_z,L_1^2\}$. Since $\mu<1$, and $x_i=\mathbb{T}^{-1}(d_i)$, we have $||x_{t-N}-\hat{x}^o_{t-N,t}\|^2\to 0$, exponentially. □

The motivation for choosing $\beta_i>0,\ i=0,\ldots,N$ is twofold. First, it will lead to filtering being an integral part of the state estimator, see Alessandri et al. [2003] for a discussion of the filtering effect in the linear MHE case. Second, it will allow us to more systematically handle cases when data are not N-exciting, as described below and in section 4. With $\beta_i>0,\ i=0,\ldots,N$ we see from Theorem 1 that the weights W_t can not be chosen arbitrarily small since the plant dynamics could in general be open loop unstable such that convergence of the observer error relies on strong enough feedback from the output in order to stabilize the error dynamics.

4. TUNING AND REGULARIZATION WITHOUT PERSISTENCE OF EXCITATION

From Theorem 1, $q_{z,t}<1$ is a sufficient condition to guarantee the convergence of the proposed MHE algorithm. To obtain $q_{z,t}<1$, $p_{z,t}$ must satisfy $p_{z,t}\le\overline{\alpha}<\sqrt{1/c_h}$, where $\overline{\alpha}$ is some constant. For N-observable systems, the condition $p_{z,t}\le\overline{\alpha}$ will depend on the existence of a (not too small) $\varepsilon>0$ such that

$$\Phi_t^T(x_{t-N},\hat{x}^o_{t-N,t})\Phi_t(x_{t-N},\hat{x}^o_{t-N,t})\ge\varepsilon I>0$$

for all $t\ge 0$. This condition comes from the requirement of U_t being N-exciting at all t. Unfortunately, since $\Phi_t(x_{t-N},\hat{x}^o_{t-N,t})$ depends on the unknown x_{t-N} we cannot monitor it directly to detect if U_t is N-exciting. Instead, we have to rely on some approximation or estimate of $\Phi_t(\cdot)$. For systems that are N-detectable but not N-observable, the situation is more complicated since we want to monitor $\Phi^o_t(z_{t-N},\hat{z}^o_{t-N,t})$ (defined in terms of the generally unknown algebraic mapping G) while $\Phi_t(x_{t-N},\hat{x}^o_{t-N,t})$ (defined in terms of the known algebraic mapping H) will in general be rank-deficient and have rank no larger than the number of observable components.

If it is assumed that $||e_{t-N}||$ is small, then

$$\Phi_t(x_{t-N},\hat{x}^o_{t-N,t})\approx\Phi_t(\hat{x}^o_{t-N,t},\hat{x}^o_{t-N,t})=\frac{\partial H}{\partial x}(\hat{x}^o_{t-N,t},U_t)=\hat{\Phi}_{x,t}.$$

Consider a singular value decomposition (SVD)

$$\hat{\Phi}_{x,t}=U_tS_tV_t^T. \tag{15}$$

Any singular value (diagonal elements of the matrix S_t) that is zero or close to zero indicates that a component is unobservable or the input is not N-exciting. Moreover, the corresponding row of the V_t matrix will indicate which components cannot be estimated. The Jacobian has the structural property that its rank will be no larger than $\dim(z)=n_z\le n_x$, due to certain components being unobservable. The N-excitation of data may therefore be monitored through the robust computation of the rank of the Jacobian matrix using the SVD (Golub and van Loan [1983]).

We know that the convergence depends on W_t being chosen such that $p_{z,t}<\sqrt{1/c_h}$. To pursue this objective, we propose to choose W_t such that, whenever possible,

$$\|(W_t\Phi_t(\hat{x}^o_{t-N,t},\hat{x}^o_{t-N,t}))^+\|=\alpha, \tag{16}$$

where $\alpha>0$ is a scalar. In (16) we have replaced the first argument x_{t-N}, which is unknown, with its estimate $\hat{x}^o_{t-N,t}$, which is a valid approximation for small e_{t-N}. In order to give zero weight on data for components that are either unobservable or unexcited, we modify this ideal design equation into the more practical and realistic design objective

$$\|(W_tU_tS_tV_t^T)^+\|=\begin{cases}\alpha, & \text{if } ||S_t||\ge\delta\\ 0, & \text{otherwise}\end{cases} \tag{17}$$

where $\delta>0$ is a design parameter. This leads to

$$W_t=(1/\alpha)V_tS^+_{\delta,t}U_t^T \tag{18}$$

satisfying

$$\|(W_t\Phi_t(\hat{x}^o_{t-N,t},\hat{x}^o_{t-N,t}))^+\|\le\alpha. \tag{19}$$

where the thresholded pseudo-inverse $S^+_{\delta,t}=\mathrm{diag}(1/\sigma_{1,t},\ldots,1/\sigma_{\ell,t},0,\ldots,0)$ where $\sigma_1,\ldots,\ \sigma_\ell$ are the singular values larger than some $\delta>0$ and the zeros correspond to small singular values whose inverse is set to zero (Golub and van Loan [1983]).

Theorem 2. Suppose assumptions (A1)-(A5) hold. If W_t is chosen according to (18) with δ being sufficiently small and $0<\alpha<1/(\sqrt{c_h}k_T)$, then W_t is bounded and e_t is locally exponentially convergent to zero.

Proof. Boundedness of W_t follows directly from $\alpha,\delta>0$. Since δ is sufficiently small and the data are N-exciting, we assume without loss of generality that the matrix $W_t\Phi^o_t(z_{t-N},\hat{z}^o_{t-N,t})$ has full rank. Using similar arguments as Lemma 1, it is easy to show that (13) in the proof of Theorem 1 is still valid.

$$Y-H(\hat{x}^o_{t-N,t},U_t)=\Phi_t(x_{t-N},\hat{x}^o_{t-N,t})(x_{t-N}-\hat{x}^o_{t-N,t}),$$
$$Y-G(\hat{z}^o_{t-N,t},U_t)=\Phi^o_t(z_{t-N},\hat{z}^o_{t-N,t})(z_{t-N}-\hat{z}^o_{t-N,t}).$$

To simplify, let $\Phi_{x,t}=\Phi_t(x_{t-N},\hat{x}^o_{t-N,t})$ and $\Phi_{z,t}=\Phi^o_t(z_{t-N},\hat{z}^o_{t-N,t})$. Since $Y-H(\hat{x}^o_{t-N,t},U_t)=Y-G(\hat{z}^o_{t-N,t},U_t)$,

$$\Phi_{z,t}(z_{t-N}-\hat{z}^o_{t-N,t})=\Phi_{x,t}(x_{t-N}-\hat{x}^o_{t-N,t}).$$

Define $\Phi_{T,t}(x_{t-N},\hat{x}^o_{t-N,t})=\int_0^1\frac{\partial}{\partial x}\mathbb{T}((1-s)x_{t-N}+s\hat{x}^o_{t-N,t},U_t)ds$, then

$$d_{t-N}-\hat{d}^o_{t-N,t}=\Phi_{T,t}(x_{t-N},\hat{x}^o_{t-N,t})(x_{t-N}-\hat{x}^o_{t-N,t}).$$

Also $z=\eta d$, where $\eta=[\mathbf{0}_{n_z\times(n_x-n_z)},I_{n_z\times n_z}]$. From the above, we have

$$\Phi_{z,t}\eta(d_{t-N}-\hat{d}^o_{t-N,t})=\Phi_{x,t}\Phi_{T,t}(x_{t-N},\hat{x}^o_{t-N,t})(d_{t-N}-\hat{d}^o_{t-N,t}).$$

From the above equation, we have

$$W_t\Phi_{z,t}\eta=W_t\Phi_{x,t}\Phi_{T,t}(x_{t-N},\hat{x}^o_{t-N,t})$$
$$\Rightarrow(W_t\Phi_{z,t})^+=\eta\Phi_{T,t}(x_{t-N},\hat{x}^o_{t-N,t})(W_t\Phi_{x,t})^+$$
$$\Rightarrow||(W_t\Phi_{z,t})^+||\le||\eta||\cdot||\Phi_{T,t}(x_{t-N},\hat{x}^o_{t-N,t})||\cdot||V_tS^+_{t,\delta}U_t^TW_t^+||. \tag{20}$$

It is known that $||\eta||=1$ and $||\Phi_{T,t}(x_{t-N},\hat{x}^o_{t-N,t})||\le k_T$, then

$$p_{z,t}=||(W_t\Phi_{z,t})^+||\le k_T||V_tS^+_{t,\delta}U_t^TW_t^+||\le k_T\alpha.$$

From Theorem 1, to obtain the convergent result, $q_{z,t}=c_hp^2_{z,t}$ must be less than 1. Therefore, the convergence condition $q_{z,t}<1$ is implied by $\alpha<1/(\sqrt{c_h}k_T)$. Note that (20) only holds for $||e_t||$ in a neighborhood of the origin, such that only local exponential convergence results. □

The tuning parameters with this adaptive choice of W_t are the non-negative scalars α, δ and β_i. It is worthwhile to notice that since they are scalars, a successful tuning of the observer will depend on appropriate scaling of the data and model equations. In section 5 we illustrate their effect using a simulation example.

When the data are not considered N-exciting at some time instant, then δ should be tuned such that the corresponding singular values of $\Phi_t(\cdot)$ will be less than δ such that the adaptive W_t will not have full rank. The means that the error in the corresponding state combinations will not be penalized by the first term in the criterion, and due to the second term they will be propagated by the open loop model dynamics.

Both $\beta_i \geq 0$ and $\delta \geq 0$ could be considered as regularization parameters that must be chosen carefully in order to tune the practical performance of the observer. In the ideal case with a perfect model, no noise, no disturbances and N-exciting data at all sampling instant one could choose $\delta = \beta_i = 0$. As a practical tuning guideline we propose to first choose at least $\beta_0 > 0$ in order to achieve acceptable filtering and performance with typical noise and disturbance levels for typical cases when the data are N-exciting. Second, $\delta > 0$ is chosen in order to achieve acceptable performance also in operating conditions when the data are not N-exciting.

5. EXAMPLE

Consider the following system

$$\dot{x}_1 = -2x_1 + x_2 \tag{21a}$$

$$\dot{x}_2 = -x_2 + x_3(u - w) \tag{21b}$$

$$\dot{x}_3 = 0 \tag{21c}$$

$$y = x_2 + v. \tag{21d}$$

It is clear that x_1 is not observable, but corresponds to a stable sub-system, while x_2 and x_3 are observable components. It is also clear that the observability of x_3 will depend on the excitation u, while x_2 is uniformly observable. One may think of x_3 as a parameter representing an unknown gain on the input, where the third state equation is an augmentation for the purpose of estimating this parameter.

The same observability and detectability properties hold for the discretized system with sampling interval $t_f = 0.1$. When $u = 0$ for all time, the rank of $\frac{\partial H}{\partial x}(\hat{x}^o_{t-N,t}, U_t)$ is 1. When u is white noise, the rank of $\frac{\partial H}{\partial x}(\hat{x}^o_{t-N,t}, U_t)$ is 2 almost always.

In this simulation example we choose $x_0 = [4, -7, 2], \bar{x}_0 = [3, -5.9, -1]$ and $N = 2$. The base case is defined as follows. Unless stated otherwise we use the adaptive W_t law (18) with $\alpha = 1$, $\delta = 0.1$, and $\beta_0 = 1, \beta_1 = 0, \beta_2 = 0$. The input is chosen with periods without informative data as follows: During $0 \leq t < 30t_f$, $u = 0$. During $30t_f \leq t < 60t_f$, u is discrete-time white noise. During $60t_f \leq t \leq 120t_f$, $u = 0$. In the simulation, the true system has an input disturbance with $w = 0.15$, and the model used in the MHE observer (11b) has an input disturbance with $w = 0.3$. Sequentially uncorrelated measurement noise, with independent uniformly distributed $v \in [-0.05, 0.05]$, is added to the following cases.

- Case 1: Default settings are used for the proposed work; for the alternative method, choose $W_t = 4I$ and $\beta = 1$. The simulation result is shown in Figure 1.
- Case 2: Choose $\beta_i = 0, i = 0, 1, 2$ for the proposed work; for the alternative method, choose $W_t = I$ and $\beta = 0$. The simulation result is shown in Figure 2.

The example shows that the adaptive weighting with the thresholded singular value inversion effectively freezes the unexcited parameter estimate and thereby avoids the parameter estimate drift that otherwise may result due to unmatched model error when there are no excitations. Additional regularization (filtering) is achieved by $\beta_0 > 0$ since otherwise the parameter estimation will be more dominated by noise, as shown by case 2.

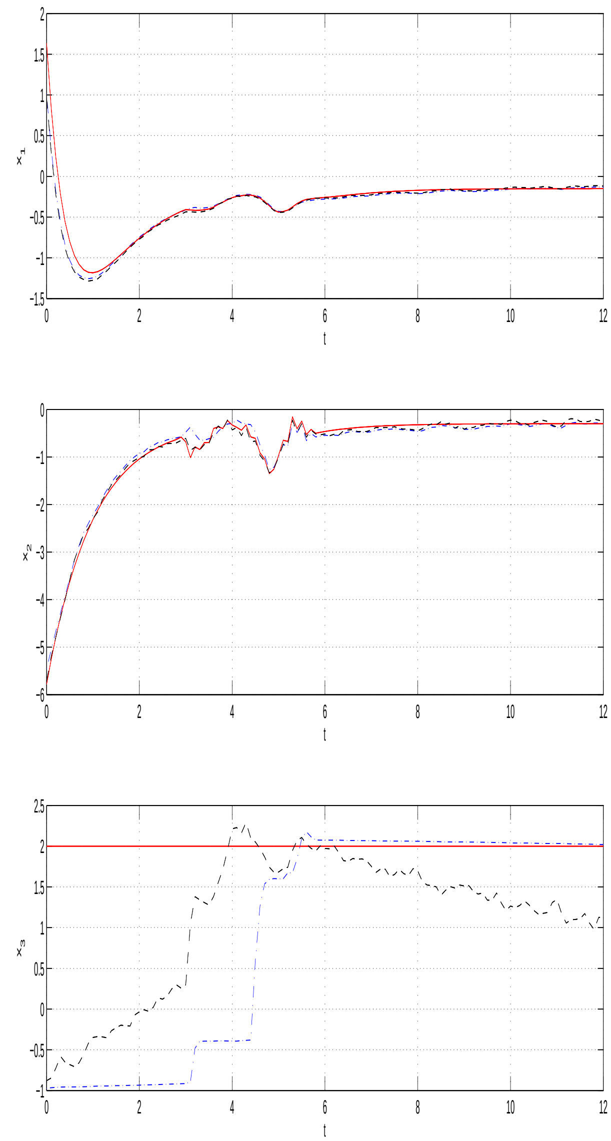

Fig. 1. Simulation results of case 1. True states are shown in solid curves; estimated states of proposed work are shown in dash-doted curves, and the alternative method is shown in dashed curves.

6. CONCLUSIONS

Theoretical and practical properties of a regularized nonlinear moving horizon observer were demonstrated in this paper. Although no convergence problems due to local minima were encountered in the simulation example in this paper, it is important to have in mind that the method will rely on a

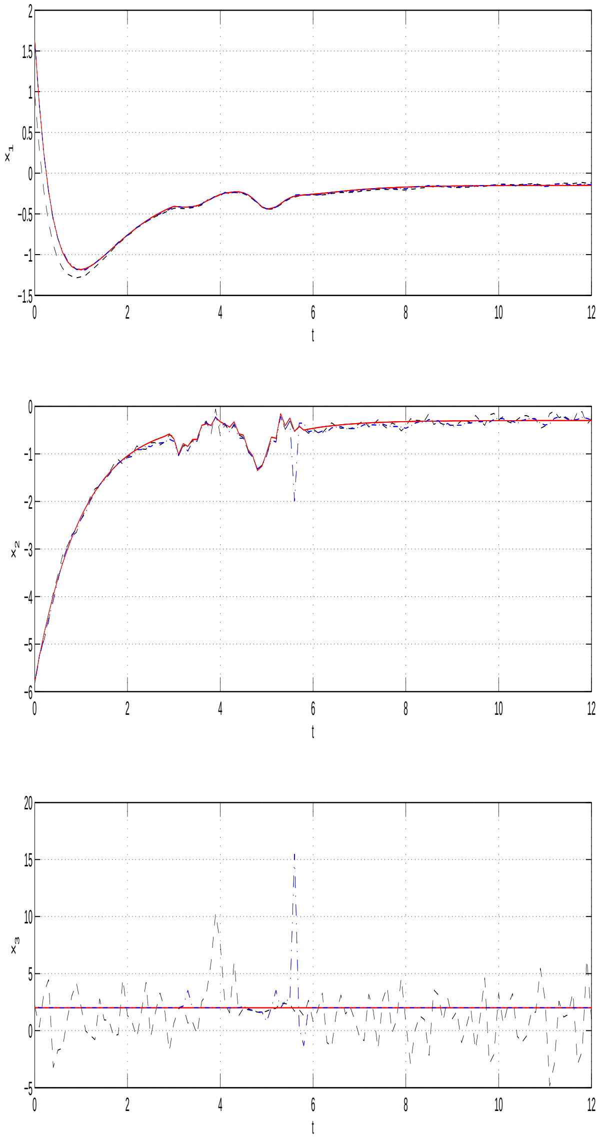

Fig. 2. Simulation results of case 2. True states are shown in solid curves; estimated states of proposed work are shown in dash-doted curves, and the alternative method is shown in dashed curves.

sufficiently accurate guess of the initial a priori estimate in cases when sub-optimal local minima exist.

The main feature of the proposed method is systematic handling of nonlinear systems that are neither uniformly observable, nor persistently excited. With the exception of the results in Moraal and Grizzle [1995b], this is to the best of the authors knowledge, an important issue not deeply studied in any other nonlinear moving horizon observer. The convergence results rely on intuitive assumptions where the key structural assumptions on the underlying system are smoothness of the functions and strong detectability.

REFERENCES

R. Abraham, J. E. Marsden, and T. Ratiu. *Manifolds, Tensor Analysis, and Applications*. Springer-Verlag New York, 1983.

M. Alamir. Optimization based non-linear observers revisited. *Int. J. Control*, 72:1204–1217, 1999.

A. Alessandri, M. Baglietto, and G. Battistelli. Receding-horizon estimation for discrete-time linear systems. *IEEE Transactions Automatic Control*, 48:473–478, 2003.

A. Alessandri, M. Baglietto, and G. Battistelli. Moving-horizon state estimation for nonlinear discrete-time systems: New stability results and approximation schemes. *Automatica*, 44: 1753–1765, 2008.

A. V. Fiacco. *Introduction to Sensitivity and Stability Analysis in Nonlinear Programming*. Academic Press, 1983.

G. H. Golub and C. F. van Loan. *Matrix computations*. Oxford University Press, 1983.

E. L. Haseltine and J. B. Rawlings. Critical evaluation of extended Kalman filtering and moving-horizon estimation. *Ind. Eng. Chem. Res*, 44:2451–2460, 2005.

G. Kreisselmeier. Adaptive observers with exponential rate of convergence. *IEEE Trans. Automatic Control*, 22:2–8, 1977.

M. Krstic, I. Kanellakopoulos, and P. Kokotovic. *Nonlinear and Adaptive Control Design*. Wiley and Sons, 1995.

P. E. Moraal and J. W. Grizzle. Observer design for nonlinear systems with discrete-time measurement. *IEEE Transactions Automatic Control*, 40:395–404, 1995a.

P. E. Moraal and J. W. Grizzle. Asymptotic observers for detectable and poorly observable systems. In *IEEE Conf. Decision and Control, New Orleans*, pages 109–114, 1995b.

E. Panteley, A. Loria, and A. Teel. Relaxed persistency of excitation for uniform asymptotic stability. *IEEE Trans. Automatic Control*, 46:1874–1886, 2001.

J. E. Parkum, N. K. Poulsen, and J. Holst. Recursive forgetting algorithms. *Int. J. Control*, 55:109–128, 1992.

T. Raff, C. Ebenbauer, R. Findeisen, and F. Allgöwer. Remarks on moving horizon state estimation with guaranteed convergence. In T. Meurer, K. Graichen, and E. D. Gilles, editors, *Control and Obsever Design for Nonlinear Finite and Infinite Dimensional Systems*, pages 67–80. Springer-Verlag, Berlin, 2005.

C. V. Rao, J. B. Rawlings, and D. Q. Mayne. Constrained state estimation for nonlinear discrete-time systems: Stability and moving horizon approximation. *IEEE Transactions Automatic Control*, 48:246–258, 2003.

K. Reif and R. Unbehauen. The extended Kalman filter as an exponential observer for nonlinear systems. *IEEE Transactions on Signal Processing*, 47:2324–2328, 1999.

K. Reif, F. Sonnemann, and R. Unbehauen. EKF-based nonlinear observer with a prescribed degree of stability. *Automatica*, 34:1119–1123, 1998.

M. E. Salgado, G. C. Goodwin, and R. H. Middleton. A modified least squares algorithm incorporating exponential resetting and forgetting. *Int. J. Control*, 47:477–491, 1988.

A. Sedoglavic. A probabilistic algorithm to test local algebraic observability in polynomial time. *J. Symbolic Computation*, 33:735755, 2002.

E. D. Sontag. A concept of local observability. *Systems and Control Letters*, 5:41–47, 1984.

A. N. Tikhonov and V. Y. Arsenin. *Solutions of Ill-posed Problems*. Wiley, 1977.