ON OPTIMIZING NONLINEAR ADAPTIVE CONTROL ALLOCATION WITH ACTUATOR DYNAMICS

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Abstract: In this work we address the optimizing control allocation problem for a nonlinear over-actuated time-varying system where parameters affine in the dynamic actuator and effector model may be assumed unknown. In-stead of finding the optimal control allocation at each time instant, a dynamic approach is considered by constructing update-laws that represent asymptotically optimal allocation search and adaptation. Using Lyapunov analysis for cascaded set-stable systems, uniform global/local asymptotic stability is guaranteed for the optimal set described by the system, the optimal allocation update-law and the adaptive update-law. Simulations of a scaled-model ship, manoeuvred at low-speed, demonstrate the performance of the proposed allocation scheme.

Keywords: Control allocation; Adaptive control; Nonlinear systems; Cascaded systems.

1. INTRODUCTION

Consider the system dynamics

$$\dot{x} = f_x(t,x) + g_x(t,x)\tau \tag{1}$$

the effector model

$$\tau := \Phi(t,x,u,\theta) := \Phi_0(t,x,u) + \Phi_\theta(t,x,u)\theta \tag{2}$$

and the actuator dynamics

$$\dot{u} = f_{u0}(t,x,u,u_{cmd}) + f_{u\theta}(t,x,u,u_{cmd})\theta \tag{3}$$

where $t \geq 0$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^r$, $\tau \in \mathbb{R}^d$, $\theta \in \mathbb{R}^m$ and $u_{cmd} \in \mathbb{R}^c$. The constant parameter vector θ contains parameters of the actuator and effector model, that will be viewed as uncertain parameters to be adapted in the low level control algorithm and used in the allocation scheme.

This work is motivated by the over-actuated control allocation problem ($d \leq r$), where the problem is described by a nonlinear system, divided into high- (1) and low-level (3) dynamics and a static interconnection (2) of the two. The main contribution of this work is to show that the static optimal control allocation problem:

$$\min_{u_d} J(t,x,u_d) \quad s.t \quad \tau_x - \Phi(t,x,u,\hat{\theta}) = 0, \tag{4}$$

where $\hat{\theta} \in \mathbb{R}^m$ is an estimate of θ, not necessarily needs to be solved exactly at each time instant.

Optimizing control allocation solutions have been derived for certain classes of over-actuated systems, such as aircraft, automotive vehicles and marine vessels, (Enns 1998, Sørdalen 1997, Bodson 2002, Luo *et al.* 2004, Luo *et al.* 2005, Poonamallee *et al.* 2005) and (Johansen *et al.* 2004). The control allocation problem is, in (Enns 1998, Sørdalen 1997) and (Bodson 2002), viewed as a *static* or *quasi-dynamic* problem that is solved independently of the dynamic control problem considering non-adaptive linear effector models of the form $\tau = Gu$, neglecting the effect of actuator dynamics. In (Luo *et al.* 2004) and (Luo *et al.* 2005) a dynamic model predictive approach is considered to solve the allocation problem with linear time-varying dynamics in the actuator model, $T\dot{u} + u = u_{cmd}$. In (Poonamallee *et al.* 2005) and (Johansen *et al.* 2004) sequential quadratic programming techniques are used to cope with nonlinearities in the control allocation problem due to singularity avoidance.

The main advantage of the control allocation approach is in general modularity and the ability to handle redundancy and constraints. In the present work we consider dynamic solutions based on the ideas presented in (Johansen 2004) and (Tjønnås and Johansen 2005). In (Johansen 2004) it was shown that it is not necessary to solve the optimization problem (4) exactly at each time instant. Furthermore a control Lyapunov function was used to derive an exponentially convergent update-law for u (related to a gradient or Newton-like optimization) such that the control allocation problem (4) could be solved dynamically. It was also shown that convergence and asymptotic optimality of the system, composed by the dynamic control allocation and a uniform globally exponentially stable trajectory-tracking controller, guarantees uniform boundedness and uniform global exponential convergence to the optimal solution of the system. In (Tjønnås and Johansen 2005) the results from (Johansen 2004) were extended by allowing uncertain parameters, associated with an adaptive law, in the effector model, and by applying set-stability analysis in order to also conclude asymptotic stability of the optimal solution.

In what follows we will extend the ideas from (Tjønnås and Johansen 2005) by introducing dynamics in the actuator model, and finally we present an example of this control allocation approach, where dynamic positioning (DP) of a model ship in different/changing weather conditions, controlled by thrusters experiencing thrust losses, is considered.

Whenever referring to the notion of set-stability, the set has the property of being nonempty, and we strictly follow the definitions given in (Tjønnås *et al.* 2006) motivated by (Teel *et al.* 2002) and (Lin *et al.* 1996).

2. ADAPTIVE CONTROL ALLOCATION WITH ACTUATOR DYNAMICS

Our results rests on the following assumptions:

Assumption 1. (Plant)

a) The states of (1) and (3) are know for all $t > t_0$.
b) There exist a continuous function $G_f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ and a constant B_g, such that $|f_x(t,x)| \leq G_f(|x|)$ and $|g_x(t,x)| \leq B_g$ for all t and x. Moreover f_x is locally Lipschitz in t and x.
c) There exist a continuous function $\varsigma_{\partial g_x} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that for all t and x, g_x is differentiable and $\left|\frac{\partial g_x}{\partial t}\right| + \left|\frac{\partial g_x}{\partial x}\right| \leq \varsigma_{\partial g_x}(|x|)$.
d) The function Φ from the static mapping (2) is twice differentiable and there exist a continuous function $G_\Phi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, such that $|\Phi_\theta| + |\Phi_0| \leq G_\Phi\left(\left|\left(x^{\mathrm{T}}, u^{\mathrm{T}}\right)^{\mathrm{T}}\right|\right)$ for all t, x, and u. Further there exist a continuous function $\varsigma_{\partial\Phi}(|x|) \geq \left|\frac{\partial \Phi_\theta}{\partial t}\right| + \left|\frac{\partial \Phi_0}{\partial t}\right| + \left|\frac{\partial \Phi_\theta}{\partial x}\right| + \left|\frac{\partial \Phi_0}{\partial x}\right| + \left|\frac{\partial \Phi_\theta}{\partial u}\right| + \left|\frac{\partial \Phi_0}{\partial u}\right|$ where $\varsigma_{\partial\Phi} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$.
e) There exists constants $\varrho_2 > \varrho_1 > 0$, such that $\forall t$, x, u and θ

$$\varrho_1 I \leq \frac{\partial \Phi}{\partial u}(t,x,u,\theta)\left(\frac{\partial \Phi}{\partial u}(t,x,u,\theta)\right)^{\mathrm{T}} \leq \varrho_2 I. \quad (5)$$

Assumption 2. (High/low level control algorithms)

a) There exists a high level control $\tau_x := k_x(t,x)$, where $|k_x(t,x)| \leq \varsigma_k(|x|)$ and $\varsigma_k : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0}$ is a continuous function, that render the equilibrium of (1) UGAS for $\tau = \tau_x$. The function k_x is differentiable, $\left|\frac{\partial k_x}{\partial t}\right| + \left|\frac{\partial k_x}{\partial x}\right| \leq \varsigma_{\partial k_x}(|x|)$, where $\varsigma_{\partial k_x} : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0}$ is continuous.
b) There exists a low-level control $u_{cmd} := k_u(t,x,u,u_d,\dot{u}_d,\hat{\theta})$ that makes the equilibrium of

$$\dot{\tilde{u}} = f_{\tilde{u}}(t,x,\tilde{u},u_d,\hat{\theta},\theta), \quad (6)$$

where $\tilde{u} := u - u_d$ and $f_{\tilde{u}}(t,x,\tilde{u},u_d,\hat{\theta},\theta) := f_{u0}(t,x,u,u_{cmd}) + f_{u\theta}(t,x,u,u_{cmd})\theta - f_d(t,x,\tilde{u},u_d,\hat{\theta})$, UGAS if $\hat{\theta} = \theta$, and if x, u_d, $\dot{u}_d$ exist for all $t > 0$. Furthermore $k_u(t,x,u,u_d,\dot{u}_d,\hat{\theta}) \leq \varsigma_u(|u - u_d|)$, where $\varsigma_u : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0}$, is a continuous differentiable function. Moreover $\left|\frac{\partial k_u}{\partial t}\right| + \left|\frac{\partial k_u}{\partial x}\right| + \left|\frac{\partial k_u}{\partial u}\right| + \left|\frac{\partial k_u}{\partial u_d}\right| + \left|\frac{\partial k_u}{\partial \dot{u}_d}\right| \leq \varsigma_{\partial k_u}(|u - u_d|)$, where $\varsigma_{\partial k_u} : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0}$ is continuous.

Remark 1. From assumption 2a) there exist a Lyapunov function $V_x : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \mapsto \mathbb{R}_{\geq 0}$ that satisfy:

$$\alpha_{x1}(|x|) \leq V_x(t,x) \leq \alpha_{x2}(|x|) \quad (7)$$

$$\frac{\partial V_x}{\partial t} + \frac{\partial V_x}{\partial x} f(t,x) \leq -\alpha_{x3}(|x|) \quad (8)$$

$$\left|\frac{\partial V_x}{\partial x}\right| \leq \alpha_{x4}(|x|), \quad (9)$$

where α_{x1}, α_{x2}, $\alpha_{x3}, \alpha_{x4} \in \mathcal{K}_\infty$, for the system $\dot{x} = f(t,x) := f_x(t,x) + g_x(t,x)k_x(t,x)$, where $g(t,x) := g_x(t,x)$. If $x(t)$, $u_d(t)$, $\dot{u}_d(t)$, and $\hat{\theta}(t) = \theta(t)$ exist for all t then there also exist a function $V_{\tilde{u}} : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and $\mathcal{K}_\infty$ functions $\alpha_{\tilde{u}1}$, $\alpha_{\tilde{u}2}$, $\alpha_{\tilde{u}3}$ such that

$$\alpha_{\tilde{u}1}(|\tilde{u}|) \leq V_{\tilde{u}}(t,\tilde{u}) \leq \alpha_{\tilde{u}2}(|\tilde{u}|) \quad (10)$$

$$\frac{\partial V_{\tilde{u}}}{\partial t} + \frac{\partial V_{\tilde{u}}}{\partial \tilde{u}} f_{\tilde{u}}(t,x(t),\tilde{u},u_d(t),\hat{\theta}(t),\theta(t)) \leq -\alpha_{\tilde{u}3}(|\tilde{u}|). \quad (11)$$

We will not discuss the details in these assumptions, but they are sufficient in order to guarantee existence of solutions and validity of the update-laws that we propose in this paper.

By assumption 2 a) we may consider a *Series Parallel* estimation model:

$$\dot{\hat{u}} = A_{\hat{u}}(u - \hat{u}) + f_{u0}(t,x,u,u_{cmd}) + f_{u\theta}(t,x,u,u_{cmd})\hat{\theta}, \quad (12)$$

where $(-A_{\hat{u}})$ is Hurwitz. This estimation model is used in a Lyapunov based indirect scheme in order to estimate the unknown but bounded parameter vector θ. For analytical purpose we use the error estimate:

$$\dot{\eta} = -A_{\hat{u}}\eta + f_{u\theta}(t,x,u,u_{cmd})\tilde{\theta} \quad (13)$$

where $\eta := u - \hat{u}$ and $\tilde{\theta} := \theta - \hat{\theta}$.

The adaptive structure of the algorithm, that we will present, is based on defining the adaptive law at the low level, incorporating the low level control algorithm and passing these parameter estimates to the control allocation algorithm. Generally we consider an adaptive control allocation approach of four modular steps where the contribution of this paper is on step three and four and the interconnection of these steps, see also Figure 1.

(1) **The high-level control algorithm.** The virtual control τ is treated as an available input to the system (1), and a virtual control law τ_x is designed such that the origin of (1) is UGAS when $\tau = \tau_x$.
(2) **The low-level control algorithm.** The actuator dynamic is controlled by the actuators through an control law u_{cmd}, such that for any smooth reference u_d, u tracks u_d asymptotically.
(3) **The control allocation algorithm (Connecting the high and low level control).** Based on the minimization problem (4) where J is a cost function that incorporates objectives such as minimum power consumption and actuator constraints (implemented as barrier functions), the Lagrangian function

$$L_{\hat{\theta}}(t,x,u_d,\tilde{u},\lambda) := J(t,x,u_d) + (k_x(t,x) - \Phi(t,x,u_d+\tilde{u},\hat{\theta}))^{\mathrm{T}}\lambda \quad (14)$$

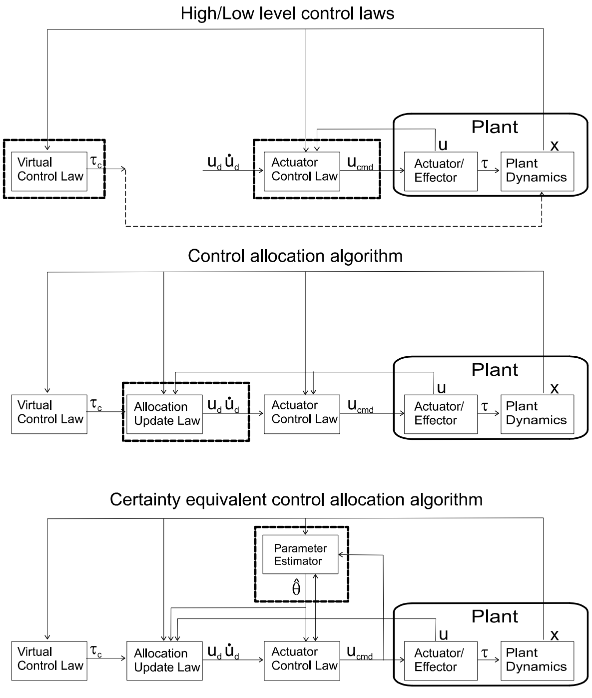

Fig. 1. Adaptive control allocation design philosophy

is introduced, and update laws for the effector reference u_d and the Lagrangian parameter λ are then defined such that u_d and λ converges to a set defined by the time-varying optimality condition.

(4) **The adaptive algorithm.** In order to cope with an possibly unknown parameter vector θ in the effector and actuator models, an adaptive law is defined. The parameter estimate is used in the control allocation scheme and a certainty equivalent adaptive optimal control allocation algorithm can be defined.

Based on Assumption 1 and 2 together with the following assumption;

Assumption 3. (Optimal control allocation)

a) The cost function $J : \mathbb{R}_{\geq t_0} \times \mathbb{R}^{n\times r} \to \mathbb{R}$ is twice differentiable and satisfies: $J(t,x,u_d) \to \infty$ as $|u_d| \to \infty$. Further there exists a continuous function $\varsigma_{\partial^2 J} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $\left|\frac{\partial^2 J}{\partial t \partial u_d}\right| + \left|\frac{\partial^2 J}{\partial x \partial u_d}\right| \leq \varsigma_{\partial^2 J}(\left|\left(x^{\mathrm T}, u_d^{\mathrm T}, \lambda^{\mathrm T}\right)\right|)\ \forall\, t,\, x,\, u_d$ and λ.

b) There exists constants $k_2 > k_1 > 0$, such that $\forall\, t$, x, $\tilde u$, $\hat\theta$ and $(u_d^{\mathrm T}, \lambda^{\mathrm T})^{\mathrm T} \notin \mathcal{O}_{u_d\lambda}$, where

$$\mathcal{O}_{u_d\lambda}(t,x) := \left\{ \left(u_d^{\mathrm T}, \lambda^{\mathrm T}\right) \in \mathbb{R}^{r+d} \Big| \left(\frac{\partial L_{\hat\theta}^{\mathrm T}}{\partial u_d}, \frac{\partial L_{\hat\theta}^{\mathrm T}}{\partial \lambda}\right) = 0 \right\},$$

then

$$k_1 I \leq \frac{\partial^2 L_{\hat\theta}}{\partial u_d^2}(t,x,u_d,\tilde u,\lambda,\hat\theta) \leq k_2 I. \qquad (15)$$

If $(u_d^{\mathrm T}, \lambda^{\mathrm T})^{\mathrm T} \in \mathcal{O}_{u_d\lambda}$, the lower bound is replaced by $\frac{\partial^2 L_{\hat\theta}}{\partial u_d^2} \geq 0$.

we formulate our main problem:

Problem: Define update-laws for u_d, $\hat\theta$ and λ such that the stability of the closed loop:

$$\Sigma_1 : \begin{cases} \dot x = f(t,x) \\ +g(t,x)\left(\Phi(t,x,u_d+\tilde u,\theta) - k_x(t,x)\right) \end{cases} \qquad (16)$$

$$\Sigma_2 : \begin{cases} \dot{\tilde u} = f_{\tilde u}(t,x,\tilde u,u_d,\hat\theta,\theta) \\ \dot u_d = f_d(t,x,\tilde u,u_d,\hat\theta) \\ \dot\lambda = f_\lambda(t,x,u_d,\hat\theta) \\ \dot\eta = -A\eta + f_{u\theta}(t,x,u,u_{cmd})\tilde\theta \\ \dot{\tilde\theta} = -f_{\hat\theta}(t,x,u_{cmd},\tilde u,u_d,\hat\theta), \end{cases} \qquad (17)$$

from (1), (2), (3), (6) and (13), is conserved and $u_d(t)$ converges to an optimal solution with respect to the minimization problem (4).

Note that system (16)-(17) takes the form of a cascade as long as $x(t)$ exists, and is viewed as a time-varying input to Σ_2, for all $t > 0$. We deal with the problem formulated by: i) defining a Lyapunov like function, $V_{u_d\lambda\tilde u\eta\tilde\theta}$, for system Σ_2 and defining explicit update-laws for u_d, λ and $\tilde\theta$ such that $\dot V_{u_d\lambda\tilde u\eta\tilde\theta} \leq 0$. ii) Further, we prove boundedness of the closed-loop system, Σ_1 and Σ_2, and use the cascade lemma (Tjønnås *et al.* 2006) to prove convergence and stability.

Based on the Lyapunov like function candidate

$$V_{u_d\lambda\tilde u\eta\tilde\theta}(t,x,u_d,\lambda,\tilde u,\eta) := V_{\tilde u}(t,\tilde u) + \frac{1}{2}\eta^{\mathrm T}\Gamma_\eta\eta + \frac{1}{2}\left(\frac{\partial L_{\hat\theta}^{\mathrm T}}{\partial u_d}\frac{\partial L_{\hat\theta}}{\partial u_d} + \frac{\partial L_{\hat\theta}^{\mathrm T}}{\partial \lambda}\frac{\partial L_{\hat\theta}}{\partial \lambda}\right) + \frac{1}{2}\tilde\theta^{\mathrm T}\Gamma_{\tilde\theta}\tilde\theta, \qquad (18)$$

we suggest the following control allocation algorithm:

$$\left(\dot u_d^{\mathrm T}, \dot\lambda^{\mathrm T}\right)^{\mathrm T} = -\Gamma\mathbb{H}_{\hat\theta}\left(\frac{\partial L_{\hat\theta}}{\partial u_d}^{\mathrm T}, \frac{\partial L_{\hat\theta}}{\partial \lambda}^{\mathrm T}\right)^{\mathrm T} - u_{ff\hat\theta} \qquad (19)$$

$$\begin{aligned} \dot{\hat\theta}^{\mathrm T} &= \Gamma_{\tilde\theta}^{-1}\left(\frac{\partial V_{\tilde u}}{\partial \tilde u} + \eta^{\mathrm T}\Gamma_\eta\right) f_{u\theta}(t,x,u,u_{cmd}), \\ &+ \Gamma_{\tilde\theta}^{-1}\left(\frac{\partial L_{\hat\theta}^{\mathrm T}}{\partial u_d}\frac{\partial^2 L_{\hat\theta}}{\partial\tilde u\partial u_d} + \frac{\partial L_{\hat\theta}^{\mathrm T}}{\partial \lambda}\frac{\partial^2 L_{\hat\theta}}{\partial\tilde u\partial\lambda}\right) f_{u\theta}(t,x,u,u_{cmd}) \\ &+ \Gamma_{\tilde\theta}^{-1}\frac{\partial L_{\hat\theta}^{\mathrm T}}{\partial \lambda}\frac{\partial^2 L_{\hat\theta}}{\partial x\partial\lambda} g(t,x)\Phi_\theta(t,x,u) \\ &+ \Gamma_{\tilde\theta}^{-1}\frac{\partial L_{\hat\theta}^{\mathrm T}}{\partial u_d}\frac{\partial^2 L_{\hat\theta}}{\partial x\partial u_d} g(t,x)\Phi_\theta(t,x,u) \end{aligned} \qquad (20)$$

where $\mathbb{H}_{\hat\theta} := \begin{pmatrix} \frac{\partial^2 L_{\hat\theta}}{\partial u_d^2} & \frac{\partial^2 L_{\hat\theta}}{\partial\lambda\partial u_d} \\ \frac{\partial^2 L_{\hat\theta}}{\partial u_d\partial\lambda} & 0 \end{pmatrix}$, Γ is a possibly time-varying symmetric positive definite weighting matrix, and $u_{ff\hat\theta} := \mathbb{H}_{\hat\theta}^{-1} u_F$ is a feed-forward like term, where:

$$\begin{aligned} u_F :=& \left(\frac{\partial^2 L_{\hat\theta}}{\partial t\partial u_d}^{\mathrm T}, \frac{\partial^2 L_{\hat\theta}}{\partial t\partial \lambda}^{\mathrm T}\right)^{\mathrm T} + \left(\frac{\partial^2 L_{\hat\theta}}{\partial x\partial u_d}^{\mathrm T}, \frac{\partial^2 L_{\hat\theta}}{\partial x\partial \lambda}^{\mathrm T}\right)^{\mathrm T} f(t,x) \\ &+ \left(\frac{\partial^2 L_{\hat\theta}^{\mathrm T}}{\partial x\partial u_d}, \frac{\partial^2 L_{\hat\theta}^{\mathrm T}}{\partial x\partial \lambda}\right)^{\mathrm T} g(t,x)(k_x(t,x) - \Phi(t,x,u_d+\tilde u,\hat\theta)) \\ &+ \left(\frac{\partial^2 L_{\hat\theta}^{\mathrm T}}{\partial \tilde u\partial u_d}, \frac{\partial^2 L_{\hat\theta}^{\mathrm T}}{\partial \tilde u\partial \lambda}\right)^{\mathrm T} f_{\tilde u}(t,x,\tilde u,u_d,\hat\theta,\theta) + \left(\frac{\partial^2 L_{\hat\theta}^{\mathrm T}}{\partial \hat\theta\partial u_d}, \frac{\partial^2 L_{\hat\theta}^{\mathrm T}}{\partial \hat\theta\partial \lambda}\right)^{\mathrm T} \dot{\hat\theta}. \end{aligned}$$

if $\det(\mathbb{H}) \neq 0$and $u_{ff\hat\theta} := 0$ if $\det(\mathbb{H}) = 0$. Hence the time derivative of $V_{u_d\lambda\tilde u\eta\tilde\theta}$ along the trajectories of (1), (3), (6), (19) and (20) is:

$$\begin{aligned} V_{u_d\lambda\tilde u\eta\tilde\theta} &= -\eta^{\mathrm T}\Gamma_\eta A\eta - \alpha_{\tilde u 3}(|\tilde u|) \\ &- \left(\frac{\partial L}{\partial u_d}^{\mathrm T}, \frac{\partial L}{\partial \lambda}^{\mathrm T}\right)\mathbb{H}_{\hat\theta}\Gamma\mathbb{H}_{\hat\theta}\left(\frac{\partial L}{\partial u_d}^{\mathrm T}, \frac{\partial L}{\partial \lambda}^{\mathrm T}\right)^{\mathrm T}. \end{aligned} \qquad (21)$$

In order to prove a global result we need to make an additional assumption on $\Phi_\theta(t,x,u)$. The assumption is based on the following lemma

Lemma 1. By Assumption 1 and (Mazenc and Praly 1996)'s lemma B.1 there exists continuous functions $\varsigma_x, \varsigma_{xu}, \varsigma_u : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, such that

$$|\Phi_\theta(t,x,u)| \leq \varsigma_x(|x|)\varsigma_{xu}(|x|) + \varsigma_x(|x|)\varsigma_u\left(\left|z_{u_d\lambda\tilde{u}\tilde{\theta}}\right|_{\mathcal{O}_{u_d\lambda\tilde{u}\tilde{\theta}}}\right)$$

where

$$\mathcal{O}_{u_d\lambda\tilde{\theta}}(t,x) := \mathcal{O}_{u_d\lambda}(t,x) \times\left\{\left(\tilde{u}^{\mathrm{T}}, \eta^{\mathrm{T}}, \tilde{\theta}^{\mathrm{T}}\right) \in \mathbb{R}^{2r+m} \middle| \left(\tilde{u}^{\mathrm{T}}, \eta^{\mathrm{T}}, \tilde{\theta}^{\mathrm{T}}\right) = 0\right\}$$

and $z_{u_d\lambda\tilde{\theta}} := \left(u_d^{\mathrm{T}}, \lambda^{\mathrm{T}}, \tilde{u}^{\mathrm{T}}, \eta^{\mathrm{T}}, \tilde{\theta}^{\mathrm{T}}\right)^{\mathrm{T}}$.

Assumption 2. (Continued)

c) There exists a $\mathcal{K}_\infty$ function $\alpha_k : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, such that

$$\alpha_k^{-1}(|x|)\alpha_{x3}(|x|) \geq \alpha_{x4}(|x|)\varsigma_{x\max}(|x|), \quad (22)$$

where $\varsigma_{x\max}(|x|) := \max(1, \varsigma_x(|x|), \varsigma_x(|x|)\varsigma_{xu}(|x|))$.

Proposition 1. If the assumptions 1, 2 and 3 are satisfied, then the solution of the closed-loop (16)-(17) is bounded with respect to a set $\mathcal{O}_{xu_d\lambda\tilde{\theta}}(t) := \mathcal{O}_{u_d\lambda\tilde{\theta}}(t,0) \times \{x \in \mathbb{R}^n \,|x = 0\}$. Furthermore $\mathcal{O}_{xu_d\lambda\tilde{\theta}}(t)$ is UGS with respect to the system (16)-(17). If in addition $\bar{f}_{u\theta}(t) := f_{u\theta}(t, x(t), u(t), u_{cmd}(t))$ is Persistently Exited (PE), then $\mathcal{O}_{xu_d\lambda\tilde{\theta}}(t)$ is UGAS with respect to system (16)-(17).

PROOF. Sketch: The main steps of this proof is to prove: i) that (18) is a Lyapunov function, when $x(t)$ is assumed to be a signal that exists for all t, with respect to $\mathcal{O}_{u_d\lambda\tilde{\theta}}$, ii) boundedness and stability of the closed loop system and iii) attractivity when $\bar{f}_{u\theta}(t)$ is PE.

i) By similar arguments as in the proof of Proposition 1 (i) in (Tjønnås and Johansen 2005) it can be shown that $\mathcal{O}_{u_d\lambda\tilde{\theta}}$ is a closed forward invariant set with respect to Σ_2. Further it can be shown that system Σ_2 is finite escape time detectable through $\left|z_{u_d\lambda\tilde{\theta}}\right|_{\mathcal{O}_{u_d\lambda\tilde{\theta}}}$ and by using theorem 2.4.7 in (Abrahamson *et al.* 1988) and a similar approach as presented in the proof of Claim 3 in (Tjønnås and Johansen 2005), it can be shown that $V_{u_d\lambda\tilde{u}\eta\tilde{\theta}}(t,x,u_d,\lambda,\tilde{u},\eta)$ is radially unbounded Lyapunov function with respect to $\left|z_{u_d\lambda\tilde{\theta}}\right|_{\mathcal{O}_{u_d\lambda\tilde{\theta}}}$.

ii) By defining $v(t,x) := V_x(t,x)$, it can be shown from Assumption 2 c) and $V_{u_d\lambda\tilde{u}\eta\tilde{\theta}}$ that $|x(t)| \leq M(r)$. Then by applying the cascade lemma from (Tjønnås *et al.* 2006) the UGS result is proved.

iii) By following ideas from the proof of the main result in (Tjønnås and Johansen 2005), the integral of $\tilde{\theta}^{\mathrm{T}}\tilde{\theta}$ can be shown to be bounded by the PE assumption, furthermore from this bound the uniform attractivity of the set $\mathcal{O}_{xu_d\lambda\tilde{\theta}}$ can be shown by contradiction. ■

Proposition 1 implies that the time-varying first order optimal set $\mathcal{O}_{xu_d\lambda\tilde{\theta}}(t)$ is uniformly stable, and in addition uniformly attractive if a PE assumption is satisfied. Thus adaptive optimal control allocation is achieved asymptotically for the closed loop system (16)-(17).

Remark 2. If the unknown parameter vector θ in the effector model (2), is not the same as in the unknown parameter vector in the actuator dynamic (3), the low level adaptive control law defined in this paper may be combined with the adaptive law presented in (Tjønnås and Johansen 2005). The analysis of this case will be considered in future work.

3. EXAMPLE

In this section simulation results of an over-actuated scaled-model ship, manoeuvred at low-speed, is presented. The scale model-ship is moved while experiencing static disturbances, caused by wind and current, and propellers trust losses. The propeller losses can be due to: *Axial Water Inflow, Cross Coupling Drag, Thruster-Hull* and *Thruster-Thruster Interaction* (see (Sørensen *et al.* 1997) and (Fossen and Blanke 2000) for details). But in this example we limit our study to thruster loss caused by *Thruster-Hull interaction*. A 3DOF horizontal plane model described by:

$$\begin{aligned} \dot{\eta}_e &= R(\psi_p)\nu \\ \dot{\nu} &= -M^{-1}D\nu + M^{-1}(\tau + b) \\ \tau &= \Phi(u,\theta), \end{aligned} \quad (23)$$

is considered, where $\eta_e := (x_e, y_e, \psi_e)^{\mathrm{T}} := (x_p - x_d, y_p - y_d, \psi_p - \psi_d)^{\mathrm{T}}$ is the north and east positions and compass heading deviations. Subscript p and d denotes the actual and desired states. $\nu := (\upsilon_x, \upsilon_y, r)^{\mathrm{T}}$ is the body-fixed velocities in surge, sway and yaw, τ is the generalized force vector, $b := (b_1, b_2, b_3)^{\mathrm{T}}$ is a static disturbance and $R(\psi_p)$ is the rotation matrix function between the body fixed and the earth fixed coordinate frame. The example we present here is based on (Lindegaard and Fossen 2003), and is also studied in (Johansen 2004) and (Tjønnås and Johansen 2005). In the considered model there are five force producing devices; the two main propellers aft of the hull, in conjunction with two rudders, and one tunnel thruster going through the hull of the vessel. ω_i denotes the propeller angular velocity and δ_i denotes the rudder deflection. $i = 1, 2$ denotes the aft actuators, while $i = 3$ denotes the tunnel thruster. Eq. (23) can be rewritten in the form of (1) and (2) by:

$$x := (\eta_e, \nu)^{\mathrm{T}}, \theta := (\theta_1, \theta_2, \theta_3)^{\mathrm{T}}, \tau := (\tau_1, \tau_2, \tau_3)^{\mathrm{T}}$$

$$u := (\omega_1, \omega_2, \omega_3, \delta_1, \delta_2)^{\mathrm{T}}, \; f_x := \begin{pmatrix} R(\psi_e + \psi_d)\nu \\ -M^{-1}D\nu + M^{-1}b \end{pmatrix},$$

$$\Phi(\nu,u,\theta) := G_u(u)\begin{pmatrix} T_1(\upsilon_x, \omega_1, \theta_1) \\ T_2(\upsilon_x, \omega_2, \theta_2) \\ T_3(\upsilon_x, \upsilon_y, \omega_3, \theta_3) \end{pmatrix}, \; g_x := \begin{pmatrix} 0 \\ M^{-1} \end{pmatrix},$$

$$G_u(u) := \begin{pmatrix} (1-D_1) & (1-D_2) & 0 \\ L_1 & L_2 & 1 \\ \Phi_{\theta 31} & \Phi_{\theta 32} & l_{3,x} \end{pmatrix}$$

$$\Phi_{3i}(u) := -l_{i,y}(1 - D_i(u) + l_{i,x}L_i(u)).$$

The truster forces are given by:

$$T_i(\upsilon_x, \omega_i, \theta_i) := T_{ni}(\omega_i) - \phi_i(\omega_i, \upsilon_x)\theta_i \qquad (24)$$

$$T_{ni}(\omega_i) := \begin{cases} k_{Tp_i}\omega_i^2 & \omega_i \geq 0 \\ k_{Tn_i}\,|\omega_i|\,\omega_i & \omega_i < 0 \end{cases},$$

$$\phi_i(\omega_i, \upsilon_x) = \omega_i \upsilon_x, \quad \phi_3(\omega_3) := \sqrt{(\upsilon_x^2 + \upsilon_y^2)}\,|\omega_3|\,\omega_3$$

$$\theta_i = \begin{cases} k_{T\theta_i}(1-w) & \upsilon_x \geq 0 \\ k_{T\theta_i} & \upsilon_x < 0 \end{cases}, \quad \theta_3 := k_{T\theta_3}$$

where $0 < w < 1$ is the wake fraction number, $\phi_i(\omega_i, \upsilon_x)\theta_i$ is the thrust loss due to changes in the advance speed, $\upsilon_a = (1-w)\upsilon_x$, and the unknown parameters θ_i represents the thruster loss factors dependent on whether the hull invokes on the inflow of the propeller or not. The rudder lift and drag forces are projected through:

$$L_i(u) := \begin{cases} (1+k_{Ln_i}\omega_i)(k_{L\delta 1_i} + k_{L\delta 2_i}\,|\delta_i|)\delta_i & ,\omega_i \geq 0 \\ 0 & ,\omega_i < 0 \end{cases},$$

$$D_i(u) := \begin{cases} (1+k_{Dn_i}\omega_i)(k_{D\delta 1_i}\,|\delta_i| + k_{D\delta 2_i}\delta_i^2) & ,\omega_i \geq 0 \\ 0 & ,\omega_i < 0 \end{cases}.$$

Furthermore it is clear from (24) that $\Phi(\nu, u, \theta) = G_u(u)Q(u) + G_u(u)\phi(\omega, \upsilon_x)\theta$, where $\phi(\omega, \upsilon_x) := diag(\phi_1, \phi_2, \phi_3)$ and $Q(u)$ represents the nominal propeller thrust.

The actuator error dynamic for each propeller is based on the propeller model presented in (Pivano *et al.* 2007) and given by

$$J_{mi}\dot{\tilde{\omega}}_i = -k_{fi}\left(\tilde{\omega}_i + \omega_{di}\right) - \frac{T_{ni}}{a_T}(\tilde{\omega}_i + \omega_{di})$$
$$+ \frac{\phi_i(\omega_i, \upsilon_x)\theta_i}{a_T} + u_{cmdi} - J_{mi}\dot{\omega}_{di} \qquad (25)$$

where $\tilde{\omega}_i := (\omega_i - \omega_{id})$, J_m is the shaft moment of inertia, k_f is a positive coefficient related to the viscous friction, a_T is a positive model constant (Pivano *et al.* 2006) and u_{cmd} is the motor torque. By the quadratic Lyapunov function $V_{\tilde{\omega}_i} := \frac{\tilde{\omega}_i^2}{2}$ it is easy to see that the control law

$$u_{cmdi} := -K_{\omega p}\tilde{\omega}_i - \frac{\phi_i(\omega_i, \upsilon_x)\hat{\theta}_i}{a_T} + J_{mi}\dot{\omega}_{di}$$
$$+ \frac{T_{ni}(\omega_{di})}{a_T} + k_{fi}\omega_{di}. \qquad (26)$$

makes the origin of (25) UGES when $\hat{\theta}_i = \theta_i$. The error dynmaics of the rudder is given by:

$$m_i\dot{\tilde{\delta}} = a_i\left(\tilde{\delta} + \delta_{di}\right) + b_i u_{cmd\delta i} - m_i\dot{\delta}_{di} \qquad (27)$$

where $\tilde{\delta} := \delta_i - \delta_{di}$, $a_i(t)$ is a known scalar function, b_i is a known scalar parameter bounded away from zero, and the controller

$$b_i u_{cmd\delta i} := -K_\delta\tilde{\delta} - a_i(t)\left(\tilde{\delta} + \delta_{di}\right) + m_i\dot{\delta}_{di} \quad (28)$$

makes the origin of (27) UGES. The parameters for the actuator model and controllers are: $a_T = 1$, $J_{mi} = 10^{-2}$, $kf_i = 10^{-4}$, $a_i = -10^{-4}$, $b_i = 10^{-5}$, $m_i = 10^{-2}$, $K_{\omega p} = 5 \cdot 10^{-3}$ and $K_\delta = 10^{-3}$

A virtual controller τ_c that stabilizes the system (23) uniformly, globally and exponentially, for some physically limited yaw rate, is proposed in (Lindegaard and Fossen 2003) and given by

$$\tau_x := -K_i R^T(\psi_p)\xi - K_p R^T(\psi_p)\eta_e - K_d\nu, \quad (29)$$

where (23) is augmented with the integral action state, $\dot{\xi} = \eta_e$.

The cost function designed for the optimization problem, (4), is:

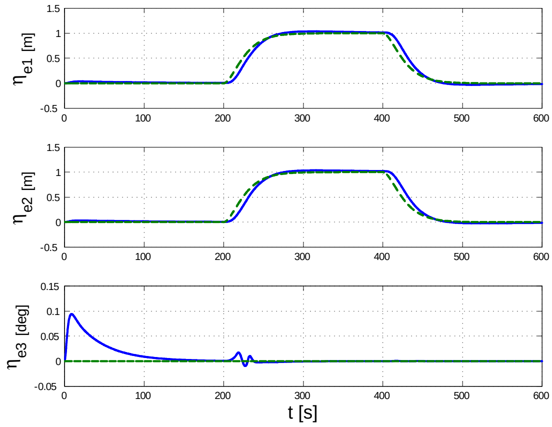

Fig. 2. Desired (dashed) and actual ship positions (solid).

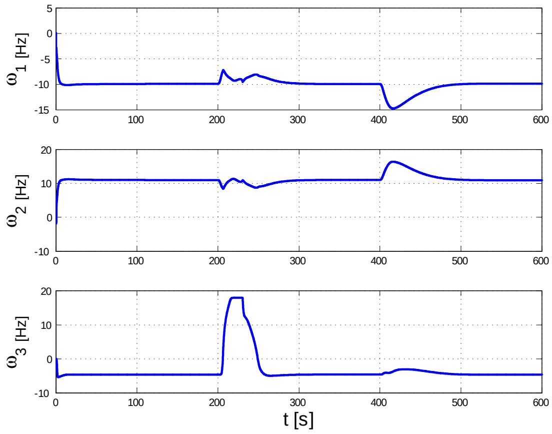

Fig. 3. Actual propeller velocities

$$J(u) := \sum_{i=1}^{3} k_i\,|\omega_i|\,\omega_i^2 + \sum_{i=1}^{2} q_i\delta_i^2 - \varsigma\sum_{i=1}^{3}\lg(-\omega_i + 18)$$
$$-\varsigma\sum_{i=1}^{3}\lg(\omega_i+18) - \varsigma\sum_{i=1}^{2}\lg(-\delta_i+35) - \varsigma\sum_{i=1}^{2}\lg(\delta_i+35),$$
$$w_\delta = 0.05, k_1 = k_2 = 0.01, k_3 = 0.02, q_1 = q_2 = 500.$$

The gain matrices are chosen as follows: $A_{\tilde{u}} := 2I_{5\times 5}$, $\Gamma_{\tilde{\theta}} := 10^{-3}$, $\Gamma_\eta := diag(10^3, 10^3, 3)$ and $\Gamma := \left(\mathbb{H}_{\hat{\theta}}^{\mathrm{T}} W \mathbb{H}_{\hat{\theta}} + \varepsilon I\right)^{-1}$ where $W := diag\left(1, 1, 1, 1, 1, 0.9, 0.9, 0.7\right)$ and $\varepsilon := 10^{-9}$. The weighting matrix W is defined such that the deviation of $\left|\frac{\partial L_{\hat{\theta}}}{\partial \lambda}\right| = \left|k(t,x) - \Phi(t,x,u,\hat{\theta})\right|$ from zero is penalized more then the deviation of $\left|\frac{\partial L_{\hat{\theta}}}{\partial u}\right|$ from zero in the search direction.

The static disturbance vector is $b := 0.05(1,1,1)^{\mathrm{T}}$, and the thruster loss vector θ is given in Figure 6.

The simulation results are presented in the Figures 2-7. The control objective is satisfied and the commanded virtual controls are tracked closely by the forces generated by the adaptive control allocation law: see Figure 5. Note that there are some deviations since ω saturates at ca. $220s$ and $420s$. Also note that the parameter estimates only converge to the true values when the ship is moving and the thrust loss is not zero. The simulations are carried out in a discrete MATLAB environment with a sampling rate of $20Hz$.

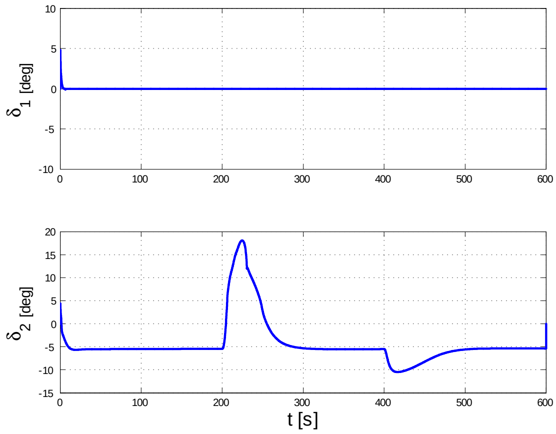

Fig. 4. Actual rudder deflection

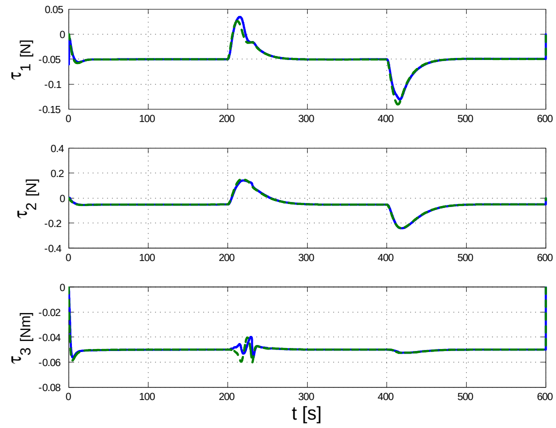

Fig. 5. The virtual control (dashed) and actual (solid) forces generated by the actuators

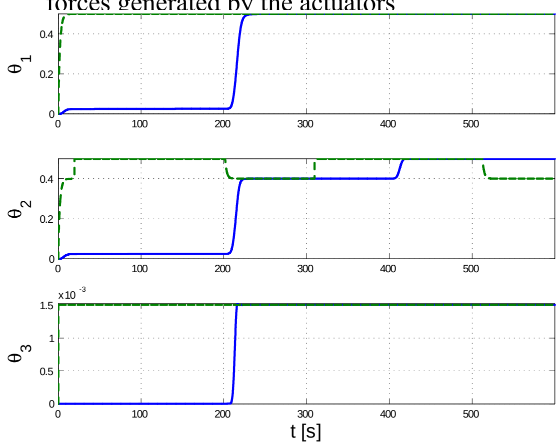

Fig. 6. Actual (dashed) and estimated (solid) loss parameters

4. ACKNOWLEDGEMENT

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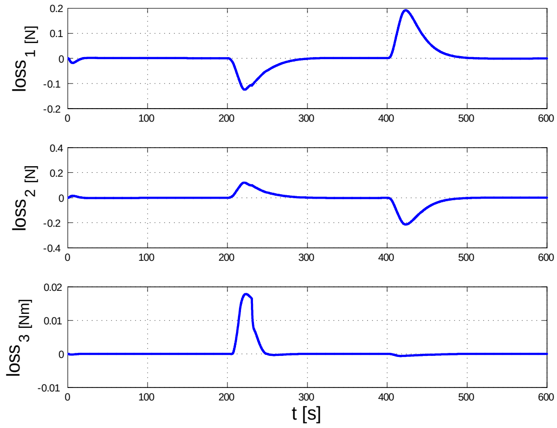

Fig. 7. Actual thrust loss

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