# Computation of Lyapunov Functions for Smooth Nonlinear Systems using Convex Optimization

Tor A. Johansen

Department of Engineering Cybernetics,

Norwegian University of Science and Technology,

N-7491 Trondheim, Norway.

Email: Tor.Arne.Johansen@itk.ntnu.no

December 14, 1999

**Abstract**

It is shown that for smooth nonlinear systems conditions for the existence of a Lyapunov function that guarantees uniform exponential stability can be formulated as linear inequalities defined pointwise in the state-space when assuming a general linearly parameterized class of smooth non-quadratic Lyapunov-function candidates. Hence, computation of the Lyapunov function involves the solution of a convex large-scale optimization problem using linear or quadratic programming. The optimization criterion can for example be selected to find a Lyapunov function which predicts fast decay rate or large region of attraction. Analysis of the tradeoff between accuracy and computational complexity as well as possible conservativeness of the procedure is given particular attention. The procedure is illustrated using numerical examples.

# 1 Introduction

This work describes a procedure for computing a Lyapunov function (if one exists) for the equilibrium point $x = 0$ for the class of non-autonomous non-linear systems

$$\dot{x} = f(x, \theta) \tag{1}$$

where $x \in R^n$ is the state vector, $\theta \in R^d$ is a possibly time-varying parameter vector, $f(0, \theta) = 0$ for all $\theta$ and $f$ is smooth. Conditions for uniform exponential stability of the equilibrium of the system (1) are considered. In other words, let $X^0 \subset R^n$ be a compact and connected region of the state space such that the origin is an interior point in $X^0$, and $\Theta \subset R^d$ a compact region of the parameter space. A Lyapunov function is sought that guarantees that if $x(0)$ is in the region of attraction $X \subset X^0$ then $x(t) \to 0$ as $t \to \infty$ at an exponential rate for any parameter trajectory that satisfies $\theta(t) \in \Theta$ for all $t$. No knowledge of $\theta$ is assumed to be available.

This paper presents a computational approach for searching for a Lyapunov function. By assuming a linear parameterization of the set of Lyapunov-function candidates, the existence of a Lyapunov function leads to two linear inequalities that must hold for every $x \in X^0$ and $\theta \in \Theta$. By discretizing the compact sets $X^0$ and $\Theta$, the possible Lyapunov functions within the selected class of candidates are characterized (approximately due to the discretization) by a finite number of linear inequalities.

Introducing a parameterized Lyapunov-function is certainly not a new idea. Lyapunov functions of polynomial form were suggested in (Zelentsovsky 1994). The problem of absolute stability for systems with nonlinearities that satisfies certain sector conditions has been studied using quadratic Lyapunov functions (Kamenetskii and Pyatnitskii 1987, Pyatnitskii and Skordodinskii 1987) as well as piecewise quadratic Lyapunov functions (Molchanov and Pyatnitskii 1986, Molchanov 1987). Recently, the class of LPV (linear parameter-varying) systems of the form $\dot{x} = A(\theta)x$ has been studied extensively using quadratic Lyapunov functions (Boyd *et al.* 1994) or $\theta$-dependent quasi-quadratic Lyapunov functions (Gahinet *et al.* 1996, Wu *et al.* 1996, Watanabe *et al.* 1996) that can be determined by convex optimization by solving linear matrix inequalities (LMIs). However, embedding a class of nonlinear systems into the LPV framework will lead to conservativeness since the nonlinear structure is not fully utilized in the Lyapunov function. In (Johansson and

Rantzer 1998, Petterson and Lennartson 1997) it is shown how piecewise quadratic Lyapunov functions can be characterized by LMIs, for general classes of nonlinear and hybrid systems. A constructive algorithm where a polyhedral region of attraction is built sequentially using a norm-based Lyapunov-function was suggested in (Brayton and Tong 1980). Extensions to piecewise linear systems were given in (Ohta *et al.* 1993). Polyhedral Lyapunov-functions might be computed using linear programming as discussed in (Blanchini 1995), and smoothed polyhedral functions were considered in (Blanchini and Miani 1996). Furthermore, another class of piecewise linear Lyapunov-functions were suggested in (Julian *et al.* 1999), where it was also shown that the resulting problem can be solved by linear programming.

The main contribution of the present work is the utilization of a flexible and general smooth parameterization of the Lyapunov-function candidates that does not introduce significant conservativeness and allows the problem to be reduced to a convex optimization problem involving linear inequality constraints at each point in the state-space. By discretization of the state-space this leads to a computational procedure based on linear or quadratic programming that is effective for systems of sufficiently low order. Furthermore, the method is flexible in the sense that it allows various additional objectives to be optimized, such as maximizing the decay rate, maximizing the region of attraction or minimizing the complexity of the Lyapunov functions. Finally, the procedure is accompanied by an analysis of the effect of discretization.

The remaining of this paper is organized as follows: In section 2 a quite general linearly parameterized set of Lyapunov functions is introduced and an infinite number of linear inequality conditions on the parameters that ensure uniform exponential stability are derived. A procedure for reducing the infinite number of linear inequalities to a finite number is described in section 3, and a numerical example is included to illustrate the computational procedure. The effect of this approximation in terms of accuracy is analysed in section 4. Computational complexity and possible conservativeness of the approach are also discussed. Some concluding remarks are given in section 4.1.

Some notation: For a vector $x \in R^n$ and any $p \geq 1$ the $p$-norm is defined by $||x||_p = (\sum_{i=1}^{n} |x_i|^p)^{1/p}$. In the limit $||x||_\infty = \max_i |x_i|$. For a matrix $A$ we define the induced norm $||A||_p = \max_{||x||_p=1} ||Ax||_p$. For a square matrix $A$, $\underline{\sigma}(A)$ and $\overline{\sigma}(A)$ denote its smallest and largest singular values, respectively, and $||A||_2 = \overline{\sigma}(A)$. The closed $\epsilon$-ball is defined as $B_\epsilon(z_0) = \{z \in R^n \mid \; ||z - z_0||_2 \leq \epsilon\}$ for any $\epsilon > 0$. The $L_p(Z)$ space with $Z \subset R^n$ and $p \geq 1$ is the completion of the space of all continuous functions $f : Z \to R^m$ with (finite) norm $||f||_p = \left(\int_{z\in Z} ||f(z)||_p^p dz\right)^{1/p}$. In the limit $||f||_\infty = \sup_{z\in Z} ||f(z)||_\infty$. A Sobolev $p$-norm on $Z$ is defined for any function $g \in L_p(Z)$ with abolutely continuous first derivative as $||g||_{1,p}^{sob} = ||g||_p + \left|\left|\frac{dg}{dx}\right|\right|_p$. For a compact set $X$, the set $\partial X$ denotes its boundary.

# 2 Convex Characterization of Uniform Exponential Stability

In this section a linearly parameterized set of Lyapunov functions is introduced. Conditions for uniform exponential stability are derived. Due to the linear parameterization, they appear as an infinite set of linear inequalities.

## 2.1 Parameterization of the Lyapunov function candidates

Consider a Lyapunov function candidate $V : X^0 \to R$ of the form

$$V(x) \quad = \quad x^T P(x) x \tag{2}$$

where the matrix valued function $P : X^0 \to R^{n\times n}$ is defined by the following linear parameterization

$$P(x) \quad = \quad \sum_{i=1}^{N} P_i \rho_i(x) \tag{3}$$

where $\rho_i : X^0 \to R$ are smooth basis-functions for all $i = 1, 2, ..., N$ and $P_1, P_2, ..., P_N$ are parameter matrices. The attention is restricted to positive semi-definite basis-functions that form a partition-of-unity:

$$\sum_{i=1}^{N} \rho_i(x) = 1, \qquad \text{for all } x \in X^0 \tag{4}$$

For later reference, the set of functions $V$ defined by (2)-(4) and fixed basis-functions $\rho_1, \rho_2, ..., \rho_N$ is denoted

$$\mathcal{V}_N \quad = \quad \left\{ x \mapsto \sum_{i=1}^{N} x^T P_i x \rho_i(x) \;\middle|\; P_1, P_2, ..., P_N \in R^{n\times n} \right\} \tag{5}$$

In the procedure developed below, we will make no further assumptions on the set of basis-functions. However, as we shall see later, in order to argue something about non-conservativeness of the approach, one may assume that they are selected from a complete basis (in a Sobolev norm). Also notice that the $P_i$-matrices are not restricted to be symmetric in general, although this may be a useful to reduce the number of parameters without significantly reducing the representational power of the Lyapunov-function candidate parameterization. Before we proceed, we would like to introduce the class of smooth locally quadratic Lyapunov function candidates generated by normalized local Gaussian basis-functions as an example of a useful class of basis-functions.

**Example: Gaussian basis-functions**

Consider basis-functions defined as

$$\rho_i(x) = \frac{\mu_i(x)}{\sum_{j=1}^{N} \mu_j(x)} \tag{6}$$

$$\mu_i(x) = \exp\left(-\frac{1}{2}\sum_{k=1}^{n}\frac{(x_k - \overline{x}_{i,k})^2}{s_{i,k}^2}\right) \tag{7}$$

where $\overline{x}_{i,k}$ and $s_{i,k}$ are parameters of the basis-functions. Eq. (6) normalizes the functions $\mu_1, \mu_2, ..., \mu_N$ such that (4) is always satisfied. Taking into account the form of these basis-functions, they can be viewed as providing a smooth interpolation between $N$ locally quadratic Lyapunov functions $x^T P_i x$ each having a certain subset of $X^0$ were they are active. The parameters $\overline{x}_{i,k}$ roughly define the location of the basis functions in the set $X$, while the parameters $s_{i,k}$ define the degree of smoothness of the basis functions. Thus, the set $\overline{x}_1, ..., \overline{x}_N$ should be selected to cover $X$ in a reasonable manner, and a typical value for $s_{i,k}$ is half of the average distance between $\overline{x}_{i,k}$ and its neigbouring points $\overline{x}_{i,j}$ for $i \neq j$. Such basis-functions are used extensively for interpolating multiple locally linear models and controllers, see e.g. (Murray-Smith and Johansen 1997). The resulting set of Lyapunov-function candidates is obviously related to piecewise quadratic Lyapunov-functions as used in (Johansson and Rantzer 1998, Petterson and Lennartson 1997). A fundamental difference is that in the present approach the Lyapunov function candidates are smooth due to the smooth interpolation, while in (Johansson and Rantzer 1998) continuity is achieved by requiring additional constraints on the matrices $P_1, P_2, ..., P_N$ to be fulfilled, and in (Petterson and Lennartson 1997) the Lyapunov function may be discontinuous.

□

## 2.2 Infinite linear inequalities

Notice that all $V \in \mathcal{V}_N$ are smooth and satisfy $V(0) = 0$. To ensure that $V$ is a Lyapunov function candidate it must satisfy $V(x) > 0$ for all $x \in X^0 - \{0\}$. Here the alternative (but somewhat stricter) condition is applied

$$V(x) \geq c_1 ||x||_2^2 \quad \text{for all} \quad x \in X^0 \tag{8}$$

for some (typically small) $c_1 > 0$. In order for $V$ to be a Lyapunov function, its time-derivative along all trajectories in $X^0 - \{0\}$ must be negative. This time-derivative is given by the function $L : X^0 \times \Theta \to R$

$$\dot{V} \;\; = \;\; L(x,\theta) \;\; = \;\; f^T(x,\theta)\frac{dV}{dx}(x) \tag{9}$$

where (2)-(3) give

$$\frac{dV}{dx}(x) \;\; = \;\; \sum_{i=1}^{N} \left( x^T P_i x \frac{d\rho_i}{dx}(x) + \left(P_i^T x + P_i x\right) \rho_i(x) \right) \tag{10}$$

Eqs. (9) and (10) leads to

$$L(x,\theta) \;\; = \;\; \sum_{i=1}^{N} \left( \left( f^T(x,\theta)\frac{d\rho_i}{dx}(x) \right) x^T P_i x + f^T(x,\theta)(P_i^T + P_i) x \rho_i(x) \right) \tag{11}$$

A condition for uniform exponential stability of the equilibrium is now that there exist matrices $P_1, P_2, ..., P_N$ such that

$$L(x,\theta) \;\; \leq \;\; -\gamma V(x), \qquad \text{for all } x \in X^0 \text{ and } \theta \in \Theta \tag{12}$$

for some constant $\gamma > 0$. The following theorem proves that the conditions (8) and (12) on $P_1, P_2, ..., P_N$ ensures that $V$ is indeed a Lyapunov function.

**Theorem 1** *Let $X^0$ be a compact and connected set. Suppose $V(x) \geq c_1 ||x||_2^2$ for all $x \in X^0$ where $c_1 > 0$, and that there exists a scalar $\gamma > 0$ such that $L(x,\theta) \leq -\gamma V(x)$ for all $x \in X^0$ and $\theta \in \Theta$. Then for all parameter trajectories $\theta(t) \in \Theta$ and initial conditions $x(0) \in X \subset X^0$, the equilibrium point is uniformly exponentially stable, i.e.*

$$||x(t)||_2 \;\; \leq \;\; \sqrt{\frac{c_2}{c_1}} ||x(0)||_2 e^{-\gamma t/2} \tag{13}$$

*where* $c_2 = \max_i \overline{\sigma}(P_i)$*. The region of attraction* $X$ *is estimated by*

$$X = \left\{ x \in X^0 \;\middle|\; V(x) \leq \inf_{\xi \in \partial X^0} V(\xi) \right\} \tag{14}$$

**Proof:** Since $L(x,\theta) < 0$ on $X^0$ it follows that $x(0) \in X$ guarantees $x(t) \in X^0$ for all $t \geq 0$. Now $\alpha_1(||x||_2) = c_1||x||_2^2 \leq V(x) \leq c_2||x||_2^2 = \alpha_2(||x||_2)$ and $\frac{d}{dt}V(x) \leq -\gamma V(x) = -\alpha_3(||x||_2)$. It is clear that $\alpha_1$, $\alpha_2$ and $\alpha_3$ are class $\mathcal{K}$ functions and the result follows from Theorem 4.1 and Corollary 4.2 of (Khalil 1992).

□

Notice that $L$ is in general a non-linear function of $x$ and $\theta$, but a linear function of the parameters of $V$ (the elements of the matrices $P_1, P_2, ..., P_N$). In other words, it can be represented in the form $L(x,\theta) = p^T l(x,\theta)$ where the function $l : X^0 \times \Theta \to R^m$ does not depend on the parameter vector $p \in R^m$ defined by $p = \left(P_1^{1,1}, P_1^{1,2}, ..., P_1^{1,n}, P_1^{2,1}, P_1^{2,2}, ..., P_N^{n,n}\right)^T$ and $P_i^{j,k}$ is the $(j,k)$-element of the $P_i$ matrix. This parameter vector has $m = Nn^2$ elements, and the function $l$ can be easily derived from (11). Furthermore, $V$ has a similar linear parametric representation $V(x) = p^T v(x)$ where the function $v : X^0 \to R^m$ can be derived easily from (2) and (3). Hence, the conditions (8) and (12) for exponential stability can be written as linear inequalities in the parameters $p$:

$$p^T v(x) \geq c_1||x||_2^2, \qquad \text{for all } x \in X^0 \tag{15}$$

$$p^T(l(x,\theta) + \gamma v(x)) \leq 0, \qquad \text{for all } x \in X^0 \text{ and } \theta \in \Theta \tag{16}$$

The constraints (15) and (16) are state- and parameter-dependent which implies that there is an infinite number of them. This leads to a so-called semi-infinite programming problem, e.g (Tanaka *et al.* 1988, Polak 1997). In section 3, finite discretizations of the state and parameter spaces are introduced in order to reduce this infinite number of linear inequalities to a finite number of linear inequalities at the cost of an approximation. The effect of this approximation is analysed in section 4.1 and related to some characteristic parameters of the system.

## 2.3 Convex objective functions and constraints

The linear inequalities (15) and (16) characterize a convex subset (the convex hull of the linear inequalities) of the parameter space $R^m$ of $V$ that define Lyapunov functions for the system (1).

Assuming this set is non-empty (at least one Lyapunov function exists), then even for very simple parameterizations of $\mathcal{V}_N$ (i.e. small $N$) there will typically exist an infinite number of Lyapunov functions. Additional objectives may thus be specified in order to find a Lyapunov function with some desirable properties. The set of Lyapunov functions will depend on the required decay rate $\gamma$, and the set of Lyapunov functions constrained by (15) and (16) will in general decrease as $\gamma$ increases until the point where no Lyapunov function within the class $\mathcal{V}_N$ exists. Typically, one will attempt to find the "best" Lyapunov function in some sense, for example the one with largest region of attraction or fastest decay rate. It is also frequently desirable to find the simplest possible Lyapunov function. This can be acheived within this framework by specifying a convex objective function that should be minimized subject to the constraints (15) and (16). Below, we will formulate linear and quadratic objective functions corresponding to these objectives. They can be selected individually or combined in a multi-objective optimization.

#### 2.3.1 Simple Lyapunov function objective

The objective of finding the simplest possible Lyapunov function can be formulated as follows. With the selected parameterization $\mathcal{V}_N$, it is natural to think of the set of quadratic Lyapunov functions as the simplest possible ones, corresponding to a constant function $P$, i.e. $P_1 = P_2 = ... = P_N$. Hence, a natural objective would be to seek matrices $P_1, P_2, ..., P_N$ that are as similar as possible. Mathematically, this is captured by the objective

$$\tilde{\phi}(p) = \sum_{i=1}^{N}\sum_{l=1}^{N}\sum_{j=1}^{n}\sum_{k=1}^{n}(P_i^{j,k} - P_l^{j,k})^2 w_{i,l} \tag{17}$$

where $w_{i,l}$ is some positive weight that in the simplest case is equal to one for all $(i, l)$, but may in general be tuned to reflect the topology of the basis-functions. It is easy to see that (17) can be written in the quadratic form $\tilde{\phi}(p) = p^T Q p$ for some positive definite matrix $Q$. Hence, this objective leads to a convex quadratic programming problem.

#### 2.3.2 Average decay rate objective and constraints

The objective of determining the Lyapunov function with the least conservative bound on the average decay rate can be formulated as maximization of $\gamma > 0$, which can be implemented as a simple line search subject to feasibility of the constraints (15)-(16). Specification of a fixed $\gamma$ is similar to a constraint on the acceptable minimum decay rate predicted by the Lyapunov function. An alternative is to minimize the average value of $\dot{V}$ over $X^0 \times \Theta$, namely

$$\tilde{\phi}(p) = \int_{X^0 \times \Theta} L(x,\theta) dx d\theta \tag{18}$$

i.e. the linear objective $\tilde{\phi}(p) = p^T c$ where $c = \int_{X^0 \times \Theta} (l(x,\theta) + \gamma v(x)) dx d\theta$. This leads to a linear programming problem.

#### 2.3.3 Region of attraction objective and constraints

Suppose we impose the following linear constraints

$$p^T v(x) \geq 1 \qquad \text{for all } x \in X^0 - X^b$$

$$p^T v(x) \leq 1 \qquad \text{for all } x \in X^a$$

where $X^a \subset X^b \subset X^0$. The motivation is that V is enforced to have a level curve $V(x) = 1$ in $X^b - X^a$ which means that it is required that the Lyapunov function predicts a region of attraction that contains $X^a$. The regions $X^a$ and $X^b$ can be defined freely. The region of attraction can be maximized by letting $X^0 - X^b$ and $X^b - X^a$ become small. If $X^a$ and $X^b$ have simple geometries such as balls or hyper-rectangles, this can be implemented easily by a line search for the maximum size of $X^a$ and $X^b$ when $X^0$ is kept fixed.

# 3 Computational Procedure

In the previous section, Lyapunov functions were characterized by an infinite number of linear inequalities. In this section they are reduced to a finite number by discretization of the state- and parameter-sets in order to make the approach computationally feasible.

## 3.1 Finite linear inequalities

The state- and parameter-dependent linear inequalities (15) and (16) define an infinite number of linear inequalities in the finite number of parameters $p$. A finite number of linear inequalities results from discretization of the compact sets $X^0$ and $\Theta$ by defining finite sets $X_d^0$ and $\Theta_d$ containing points where the following constraints on the parameters $p$ are imposed for some $\alpha > 0$

$$p^T v(x) \geq c_1 ||x||_2^2 \quad \text{and} \quad p^T(l(x,\theta) + \alpha v(x)) \leq 0, \qquad \text{for all } (x,\theta) \in X_d^0 \times \Theta_d \tag{19}$$

These inequalities are stacked in matrices as follows

$$\tilde{V}p \geq \tilde{c} \tag{20}$$

$$(\tilde{L} + \alpha\tilde{\tilde{V}})p \leq 0 \tag{21}$$

where the rows of $\tilde{V}$ corresponds to $v^T(x)$ for each $x \in X_d^0$, the elements of $\tilde{c}$ are $c_1||x||_2^2$ for each $x \in X_d^0$, the rows of $\tilde{L}$ corresponds to $l^T(x,\theta)$ for each $(x,\theta) \in X_d^0 \times \Theta_d$, and the rows of $\tilde{\tilde{V}}$ corresponds to $v^T(x)$ for each $(x,\theta) \in X_d^0 \times \Theta_d$. Additional linear constraints may be due to other objectives, such as a required region of attraction as discussed in section 2.3. Hence, (20)-(21) define a finite number of linear constraints in a finite number of variables, which are computationally feasible using standard linear or quadratic programming, e.g. (Luenberger 1989).

Because of the finite discretization, the existence of a parameter vector $p \in R^m$ that satisfies (20)-(21) is not sufficient to guarantee that $V$ is a Lyapunov function. It must be checked that there exists a $\gamma > 0$ such that $L(x,\theta) \leq -\gamma V(x)$ and $V(x) > 0$ holds for all $x \in X^0 - \{0\}$ and $\theta \in \Theta$. Then it can be argued that $x(0) \in X$ guarantees uniform exponential stability of the origin. In practise, these conditions must be checked in a sufficiently dense finite set of checking points $X_c^0 \times \Theta_c \subset X^0 \times \Theta$. If they do not hold, the density of the set of design points $X_d^0$ and the parameterization of $P$ should be made finer, and the above procedure iterated. Theoretical bounds and guidelines for selecting the number of points in $X_d^0 \times \Theta_d$ and $X_c^0 \times \Theta_c$ are given in section 4.

## 3.2 Procedure

A computational procedure for searching for a Lyapunov function is:

- Input data: The system function $f$, a compact and connected set $X^0$ that contains the origin as an interior equilibrium point, and a compact set $\Theta$.
- Step 1: Select a set of basis functions $\rho_1, \rho_2, ..., \rho_N$ for $\mathcal{V}_N$.

- Step 2: Select finite sets $X_d^0 \subset X^0$ and $\Theta_d \subset \Theta$ and possibly a minimum region of attraction $X^a \subset X^b \subset X^0$.

- Step 3: Solve the convex optimization problem of determining a feasible (or maximum) $\alpha > 0$ while minimizing one of the linear or quadratic objectives (17) or (18) subject to the linear constraints (20) and (21) with respect to $p$ (there may be additional constraints if a minium region of attraction is specified). If no solution was found, go to either Step 1 or Step 2.

- Step 4: Generate "sufficiently dense" but finite checking sets $X_c \subset X^0$ and $\Theta_c \subset \Theta$.

- Step 5: If $L(x,\theta) \leq -\gamma V(x)$ does not hold on $X_c \times \Theta_c$ or $V(x) > 0$ does not hold on $X_c$ for some $\gamma > 0$, go to either Step 1 or Step 2.

- Output data: If the procedure converges, a Lyapunov function has been found, and the the region of attraction $X$ can be estimated according to (14).

Note that the computational efficiency of this iterative procedure might be improved if the refinement of the discretization and parameterization of the Lyapunov function at the next iteration is constructed such that the optimal parameters from the previous iteration can be applied to initialize the next linear or quadratic program (Polak 1997).

## 3.3 Numerical example

Consider the autonomous non-linear system defined by

$$f(x) = \begin{pmatrix} -3x_1 + x_2 \\ \frac{2x_1^2}{0.3+(x_2+0.4)(x_2-0.6)} - 2x_2 \end{pmatrix} \tag{22}$$

We apply normalized local basis-functions of the form (6)-(7) and define the subset of the state-space $X^0 = [-1,1] \times [-1,1] \subset R^2$. Consider the following three alternative parameterizations of the Lyapunov function candidates:

1. $\mathcal{P}_1$: Quadratic Lyapunov function candidates, $N = 1$. The basis function parameters are $\overline{x}_1 = (0,0)^T$ and $s_{1,k} = 1$.

2. $\mathcal{P}_4$: Non-quadratic Lyapunov function candidates composed from four locally quadratic functions (symmertically located in $X^0$), $N = 4$. The basis function parameters are $\overline{x}_1 =$

$(-1/2,-1/2)^T$, $\overline{x}_2 = (1/2,-1/2)^T$, $\overline{x}_3 = (-1/2,1/2)^T$, $\overline{x}_4 = (1/2,1/2)^T$ and $s_{i,k} = 1/2$ for all $i,k$.

3. $\mathcal{P}_9$: Non-quadratic Lyapunov function candidates composed from nine locally quadratic functions (symmertically located in $X^0$), $N = 9$. The basis function parameters are $\overline{x}_1 = (-2/3,-2/3)^T$, $\overline{x}_2 = (0,-2/3)^T$, $\overline{x}_3 = (2/3,-2/3)^T$, $\overline{x}_4 = (-2/3,0)^T$, $\overline{x}_5 = (0,0)^T$, $\overline{x}_6 = (2/3,0)^T$, $\overline{x}_7 = (-2/3,2/3)^T$, $\overline{x}_8 = (0,2/3)^T$, $\overline{x}_9 = (2/3,2/3)^T$, and $s_{i,k} = 1/3$ for all $i,k$.

In all cases below we uniformly discretize $X^0$ with 441 points where the constraints are imposed, which leads to problems of reasonable computational complexity, with 882 inequalities (or up to 1323 inequalities if constraints on the region of attraction are also included) in 4, 16 and 36 variables, respectively.

First, assume we seek a Lyapunov function where the objective is to predict the maximum decay rate. This is impelemted by maximizing the parameter $\alpha$ subject to feasibility of the exponential stability conditions. In addition, the Lyapunov function complexity measure (17) is minimized (for each fixed $\alpha$). The results are summarized in Table 1, and the level curves of the Lyapunov functions are shown in Figure 1. Observe that the predicted decay rate increases considerably as the number of parameters in the Lyapunov function candidate parameterization increases. It can also be observed that the predicted region of attraction typically decreases when the predicted decay rate increases, which is not surprising since it is not an objective in this case. The example illustrates that a more complex Lyapunov function gives a less conservative prediction of the decay rate.

Second, suppose we want to find a Lyapunov function where the objective is to predict the maximum region of attraction within $X^0 = [-1,1] \times [-1,1]$. This is implemented by defining $X^a$ to be a square $X^a = [-r,r] \times [-r,r]$ within the square $X^0$, and $X^b = [-0.95,0.95] \times [-0.95,0.95]$. In addition, the Lyapunov function complexity measure (17) is minimized (for each fixed $r$), and a minimum decay rate $\alpha = 1$ is required. The results are summarized in Table 1, and the level curves of the Lyapunov functions are shown in Figure 1. We observe that the predicted region of attraction increases considerably as the number of parameters in the Lyapunov function candidate parameterization increases. Hence, the example illustrates that a more complex Lyapunov function gives a less conservative prediction of the region of attraction.

Finally, consider the non-autonomeous non-linear system defined by

$$f(x,\theta) \quad = \quad \begin{pmatrix} -3(1+\theta)x_1 + x_2 \\ \frac{2(1-\theta)x_1^2}{0.3+(x_2+0.4)(x_2-0.6)} - 2x_2 \end{pmatrix} \tag{23}$$

where $\theta(t) \in [-0.2, 0.2]$ is a time-varying unknown parameter. Note that in the nominal case with $\theta = 0$, this corresponds to the autonomeous system (22) considered above. Now we seek a Lyapunov function that proves exponential stability of (23) for all trajectories of $\theta(t) \in [-0.2, 0.2]$. We apply the same three alternative Lyapunov function parameterizations $\mathcal{P}_1$, $\mathcal{P}_4$ and $\mathcal{P}_9$ as above, and in addition we introduce a discretization along the $\theta$-dimension such that the number of inequalities are 4410 (or up to 6615 if constraints on the region of attraction are added). The results are summarized in Table 2, and the level curves of the Lyapunov functions are shown in Figure 2. As expected, due to the uncertain parameter $\theta$ the regions of attraction and predicted decay rates are somewhat smaller than in the nominal case. As in the nominal case, a more complex Lyapunov functions gives a less conservative prediction of the region of attraction or decay rate. Observe that for $N = 1$ and $\alpha = 1$ no (quadratic) Lyapunov function satisfying the constraints exists.

## 4 Accuracy vs. computation efficiency tradeoff

The computational procedure described above is not an algorithm, in the sense that several of the steps require further specification. In general, very dense uniform grids of design and checking points should be avoided, in particular in high-dimensional state and parameter spaces since they will make the problem computationally intractable due to a large number of inequalities.

In this section the required granularity of points in the sets $X_d^0 \times \Theta_d$ and $X_c^0 \times \Theta_c$ are investigated theoretically, and some bounds are derived. These can be useful in order to generate sets of points that guarantees the existence of a Lyapunov-function despite the finite discretization, and also in order to understand which factors influence the required granularity.

## 4.1 The effect of discretization

Important information about the required granularity of design and checking points in the state and parameter spaces can be determined by analyzing the complexity of $f$ over different regions in the state subset $X^0$. The idea is that if $f$ is a highly nonlinear function in some regions of the state or parameter spaces, a useful heuristic may be to allow large variations in $P$ in these regions, and to increase the density of design and checking points in these regions. Define the checking set granularity function $\varepsilon : X^0 \times \Theta \to R$

$$\varepsilon(x,\theta) = \inf_{(\xi,\zeta)\in X_c^0\times\Theta_c} ||(x,\theta)-(\xi,\zeta)||_2 \tag{24}$$

The usefulness of the above mentioned heuristic can be seen theoretically by assuming $f$, $\rho_i$ and $d\rho_i/dx$ to be bounded and locally Lipschitz functions in the sense that for every $(x_1,\theta_1),(x_2,\theta_2)\in B_{\varepsilon(x,\theta)}((x,\theta))$ there exist bounded functions $L_f : X^0\times\Theta\to R$, $L_\rho : X^0 \to R$ and $L_{\rho'} : X^0 \to R$ that satisfy

$$||f(x_1,\theta_1)-f(x_2,\theta_2)||_2 \leq L_f(x,\theta)||(x_1,\theta_1)-(x_2,\theta_2)||_2 \tag{25}$$

$$|\rho_i(x_1)-\rho_i(x_2)| \leq L_\rho(x)||x_1-x_2||_2 \tag{26}$$

$$\left|\left|\frac{d\rho_i}{dx}(x_1)-\frac{d\rho_i}{dx}(x_2)\right|\right|_2 \leq L_{\rho'}(x)||x_1-x_2||_2 \tag{27}$$

and define $K_1(x,\theta)=\sup_{(\xi,\zeta)\in B_{\varepsilon(x,\theta)}((x,\theta))}||f(\xi,\zeta)||_2$, $K_2(x)=\sup_{\xi\in B_{\varepsilon(x)}(x)}\left|\left|\frac{d\rho_i}{dx}(\xi)\right|\right|_2$, $\overline{P}=\max_i\overline{\sigma}(P_i)$, and $\overline{X}^0=\sup_{x,\xi\in X^0}||x-\xi||_2$.

**Theorem 2** *Suppose $X^0$ and $\Theta$ are compact sets, $V(x)>0$ for all $x\in X^0-\{0\}$ and $f$, $\rho_i$ and $d\rho_i/dx$ are bounded and locally Lipschitz functions. Let $\gamma>0$ be given and suppose there exists an $\alpha>\gamma>0$ such that for all $(\xi,\zeta)\in X_c^0\times\Theta_c$*

$$L(\xi,\zeta) \leq -\alpha V(\xi) \tag{28}$$

*Assume the checking grid granularity $\varepsilon(x,\theta)$ is so fine that*

$$\varepsilon(x,\theta) \leq (\alpha-\gamma)\frac{V(x)}{Q(x,\theta)} \tag{29}$$

*where*

$$\begin{aligned} Q(x,\theta) = {} & K_1(x,\theta)\overline{P}\left(N\overline{X}^0\left(L_{\rho'}(x)\overline{X}^0+2K_2(x)+2L_\rho(x)\right)+2\right) \\ & +\overline{X}^0\overline{P}(2+NK_2(x)\overline{X}^0)L_f(x,\theta)+\alpha\overline{PX}^0(2+NL_\rho(x)\overline{X}^0) \end{aligned} \tag{30}$$

*Then for all* $x \in X^0$ *and* $\theta \in \Theta$

$$L(x,\theta) \quad \le \quad -\gamma V(x) \tag{31}$$

**Proof:** Let $(x,\theta) \in X^0 \times \Theta$ and $(\xi,\zeta) \in (X_c^0 \times \Theta_c) \cap B_{\varepsilon(x,\theta)}((x,\theta))$ be arbitrary. From (28) we get

$$\begin{aligned} L(x,\theta) &= L(\xi,\zeta) + (L(x,\theta) - L(\xi,\zeta)) && (32)\\ &\le -\alpha V(x) + \alpha(V(x) - V(\xi)) + (L(x,\theta) - L(\xi,\zeta)) && (33)\\ &= -\gamma V(x) - (\alpha-\gamma)V(x) + \alpha(V(x) - V(\xi)) + (L(x,\theta) - L(\xi,\zeta)) && (34)\end{aligned}$$

It can be verified directly that the definition of $Q$ implies

$$L(x,\theta) \quad \le \quad -\gamma V(x) - (\alpha-\gamma)V(x) + Q(x,\theta)||(x,\theta) - (\xi,\zeta)||_2 \tag{35}$$

and from (24) it is clear that

$$L(x,\theta) \quad \le \quad -\gamma V(x) - (\alpha-\gamma)V(x) + Q(x,\theta)\varepsilon(x,\theta) \ \le \ -\gamma V(x)$$

where the last inequality follows from (29).

□

This theorem relates the required local checking set granularity to local complexity measures (local Lipschitz constants) of the system function and the basis function of the Lyapunov function. The above result only concerns the discretization of the condition $\dot{V} \le 0$. Discretization of the additional condition $V(x) \ge c_1||x||_2^2$ does usually not impose any requirements on the checking set granularity since $V(x)$ is increasing with increasing $||x||_2$, and it is only required that $V(x) > 0$ for $x \in X^0 - \{0\}$.

The lower bound on the required checking set granularity given by the right hand side of (29) is typically of limited practical usefulness. The reason for this is twofold. First, the involved parameters are typically hard to compute, and second, the bound is typically too conservative. The main importance of Theorem 2 is therefore that it proves existence of a lower bound and also that it illustrates how the lower bound depend on the characteristic parameters of the problem such as Lipschitz constants and complexity of the Lyapunov function parameterization.

## 4.2 Parameterization of the Lyapunov function candidates, revisited

The parameterization of the set of Lyapunov-function candidates $\mathcal{V}_N$ is of importance in order to determine a tight Lyapunov function with a computationally efficient procedure. In order to argue that this parameterization does not introduce significant conservativeness, it is convenient if the basis-functions are selected from a set of functions that guarantees that any smooth Lyapunov function candidate and its gradient can be approximated to arbitrary accuracy on $X^0$. By selecting the basis-functions from a basis that is complete in a Sobolev norm this can be satisfied:

**Theorem 3** *Suppose $X^0$ is a compact set and $0 \in X^0$. Let $W : X^0 \to R$ be an arbitrary smooth Lyapunov function candidate, i.e. $W(0) = 0$, $W(x) > 0$ for any $x \in X^0 - \{0\}$. Suppose the set of functions $\mathcal{G}(X^0)$ defines a complete basis for the set of smooth functions in the Sobolev p-norm defined on $X^0$, where $p \geq 1$ is arbitrary. Then for any $\delta > 0$ there exist basis-functions $\rho_1, \rho_2, ..., \rho_N \in \mathcal{G}(X^0)$ and matrices $P_1, P_2, ..., P_N$ such that $V(x) = \sum_{i=1}^N x^T P_i x \rho_i(x)$ satisfies $||W - V||_p \leq \delta$, $\left|\left|\frac{dW}{dx} - \frac{dV}{dx}\right|\right|_p \leq \delta$.*

**Proof:** Taylor's theorem (see (Abrahamson *et al.* 1988), Theorem 2.4.15 for a general version of this) gives

$$
\begin{aligned}
W(x) &= W(0) + \frac{dW}{dx}(0)x + x^T \tilde{P}(x) x & (36)\\
\tilde{P}(x) &= \int_0^1 (1-s)\left(\frac{d^2W}{dx^2}(sx) - \frac{d^2W}{dx^2}(0)\right) ds + \frac{d^2W}{dx^2}(0) & (37)
\end{aligned}
$$

Recall that $W(0) = 0$ and $dW/dx(0) = 0$ since $x = 0$ is a minimum for $W$. Hence,

$$
W(x) - V(x) = x^T(\tilde{P}(x) - P(x))x \qquad (38)
$$

and

$$
\frac{dW}{dx}(x) - \frac{dV}{dx}(x) = (\tilde{P}(x) - P(x))^T x + \left(\frac{d}{dx}\left(x^T\tilde{P}(x)\right) - \frac{d}{dx}\left(x^T P(x)\right)\right) x \qquad (39)
$$

The result follows from (38) and (39), the compactness of $X^0$ and the completeness of $\mathcal{G}(X^0)$ since $\tilde{P}$ is smooth due to the smoothness of $W$, cf. (37).

□

Notice that the set of smooth locally quadratic Lyapunov function candidates with Gaussian basis-functions introduced in section 2.1 can be shown to be complete in the Sobolev norm, cf. (Rovatti

1996).

# 5 Concluding Remarks

A numerical procedure for computation of Lyapunov functions for smooth nonlinear systems using linear or quadratic programming is suggested. The advantages of the approach is its fexibility with respect to parameterization of the smooth Lyapunov function candidates in terms of basis-functions, and flexibility to impose additional objectives and constraints on the Lyapunov functions. The applicability of the procedure is somewhat limited by the fact that the number of linear inequalities characterizing the Lyapunov functions will grow exponentially with both the dimension of the state-space and the required accuracy if one applies regular grids of design and checking points. In order to help reducing the size of the numerical problem it is shown that the design and checking points should be concentrated in regions of the state-space where the nonlinearities are most pronounced. A lower bound on the required granularity of the checking point set is characterized in terms of regularity properties of the system function and basis functions in the Lyapunov function parameterization. For the purpose of checking stability, conservativeness of the approach is due to finite parameterization of the set of Lyapunov function candidates, and finite discretization of the state-space.

# Acknowledgments

This work was sponsored by the European Comission under the ESPRIT Long Term Research project 28104 $H^2C$.

# References

Abrahamson, R., J. E. Marsden and T. Ratiu (1988). *Manifolds, Tensor Analysis and Applications (2nd Edition).* Springer-Verlag, New York.

Blanchini, F. (1995). Nonquadratic Lyapunov functions for robust control. *Automatica* **31**, 451–461.

Blanchini, F. and S. Miani (1996). A new class of universal Lyapunov functions for the control of uncertain linear systems. In: *Proc. 35th Conf. Decision and Control, Kobe, Japan.* pp. 1027–1032.

Boyd, S., L. El Ghaoui, E. Feron and V. Balahrishnan (1994). *Linear Matrix Inequalities in System and Control Theory.* SIAM, Philadelphia.

Brayton, R. K. and C. H. Tong (1980). Constructive stability and asymptotic stability of dynamical systems. *IEEE Trans. Circuits and Systems* **27**, 1121–1130.

Gahinet, P., P. Apkarian and M. Chilali (1996). Affine parameter-dependent Lyapunov function and real parametric uncertainty. *IEEE Trans. Automatic Control* **41**, 436–442.

Johansson, M. and A. Rantzer (1998). Computation of piecewise quadratic Lyapunov functions for hybrid systems. *IEEE Trans. Automatic Control* **43**, 555–559.

Julian, P., J. Guivant and A. Desages (1999). A parameterization of piecewise linear Lyapunov functions via linear programming. To appear in Int. J. Control.

Kamenetskii, V. A. and E. S. Pyatnitskii (1987). Gradient method of constructing Lyapunov functions in problems of absolute stability. *Automation and Remote Control* **48**, 1–9.

Khalil, H. K. (1992). *Nonlinear Systems.* Macmillan, New York.

Luenberger, D. G. (1989). *Introduction to Linear and Nonlinear Programming.* Addison-Wesley, Inc., Reading, MA.

Molchanov, A. P. (1987). Lyapunov functions for nonlinear discrete-time control systems. *Automation and Remote Control* **48**, 728–736.

Molchanov, A. P. and E. S. Pyatnitskii (1986). Lyapunov functions that specify necessary and sufficient conditions of absolute stability of nonlinear nonstationary control systems. parts I - III. *Automation and Remote Control* **47**, 344–354, 443–451, 620–630.

Murray-Smith, R. and Johansen, T. A., Eds.) (1997). *Multiple Model Approaches to Modelling and Control.* Taylor and Francis, London.

Ohta, Y., H. Imanishi, L. Gong and H. Haneda (1993). Computer generated Lyapunov functions for a class of nonlinear systems. *IEEE Trans. Circuits and Systems* **40**, 343–354.

Petterson, S. and B. Lennartson (1997). Exponential stability analysis of nonlinear systems using LMIs. In: *Proc. IEEE Conf. Decision and Control, San Diego.*

Polak, E. (1997). *Optimization: Algorithms and Consistent Approximation.* Springer, New-York.

Pyatnitskii, E. S. and V. I. Skordodinskii (1987). A criterion of absolute stability of nonlinear sampled-data control systems in the form of numerical procedures. *Automation and Remote Control* **48**, 1190–1198.

Rovatti, R. (1996). Takagi-Sugeno models as approximators in Sobolev norms - the SISO case. In: *Proc. IEEE Conf. Fuzzy Systems, New Orleans.* pp. 1060–1066.

Tanaka, Y., M. Fukushima and T. Ibaraki (1988). A comparative study of several semi-infinite nonlinear programming algorithms. *European Journal of Operational Research* **36**, 92–100.

Watanabe, R., K. Uchida and M. Fujita (1996). A new LMI approach to analysis of linear systems with scheduling parameter - Reduction to finite number of LMI conditions. In: *Proc. 35th Conference on Decision and Control, Kobe.* pp. 1663–1665.

Wu, F., X. H. Yang, A. Packard and G. Becker (1996). Induced $L_2$-norm control for LPV systems with bounded parameter variation rates. *Int. J. Nonlinear and Robust Control* **6**, 983–998.

Zelentsovsky, A. L. (1994). Nonquadratic Lyapunov function for robust stability analysis of linear uncertain systems. *IEEE Trans. Automatic Control* **39**, 135–138.

| Number of basis-functions | Objective | Result |
|---|---|---|
| $N = 1$ | Decay rate | $\alpha = 1.43$, $r = 0.36$ |
| $N = 4$ | Decay rate | $\alpha = 3.17$, $r = 0.20$ |
| $N = 9$ | Decay rate | $\alpha = 5.06$, $r = 0.30$ |
| $N = 1$ | Region of attraction | $\alpha = 1$, $r = 0.42$ |
| $N = 4$ | Region of attraction | $\alpha = 1$, $r = 0.52$ |
| $N = 9$ | Region of attraction | $\alpha = 1$, $r = 0.85$ |

Table 1: Summary of the properties of the computed Lyapunov functions for the autonomeous system (22). The parameter $\alpha$ is the bound on the decay rate, while $r$ is the radius of the largest square $[-r, r] \times [-r, r] \subset X^0$ containing the region of attraction $X$.

| Number of basis-functions | Objective | Result |
|---|---|---|
| $N = 1$ | Decay rate | $\alpha = 0.56$, $r = 0.30$ |
| $N = 4$ | Decay rate | $\alpha = 2.48$, $r = 0.16$ |
| $N = 9$ | Decay rate | $\alpha = 5.50$, $r = 0.13$ |
| $N = 1$ | Region of attraction | $\alpha = 1$, $r = 0$ |
| $N = 4$ | Region of attraction | $\alpha = 1$, $r = 0.30$ |
| $N = 9$ | Region of attraction | $\alpha = 1$, $r = 0.70$ |

Table 2: Summary of the properties of the computed Lyapunov functions for the non-autonomeous system (23). The parameter $\alpha$ is the bound on the decay rate, while $r$ is the radius of the largest square $[-r, r] \times [-r, r] \subset X^0$ containing the region of attraction $X$.

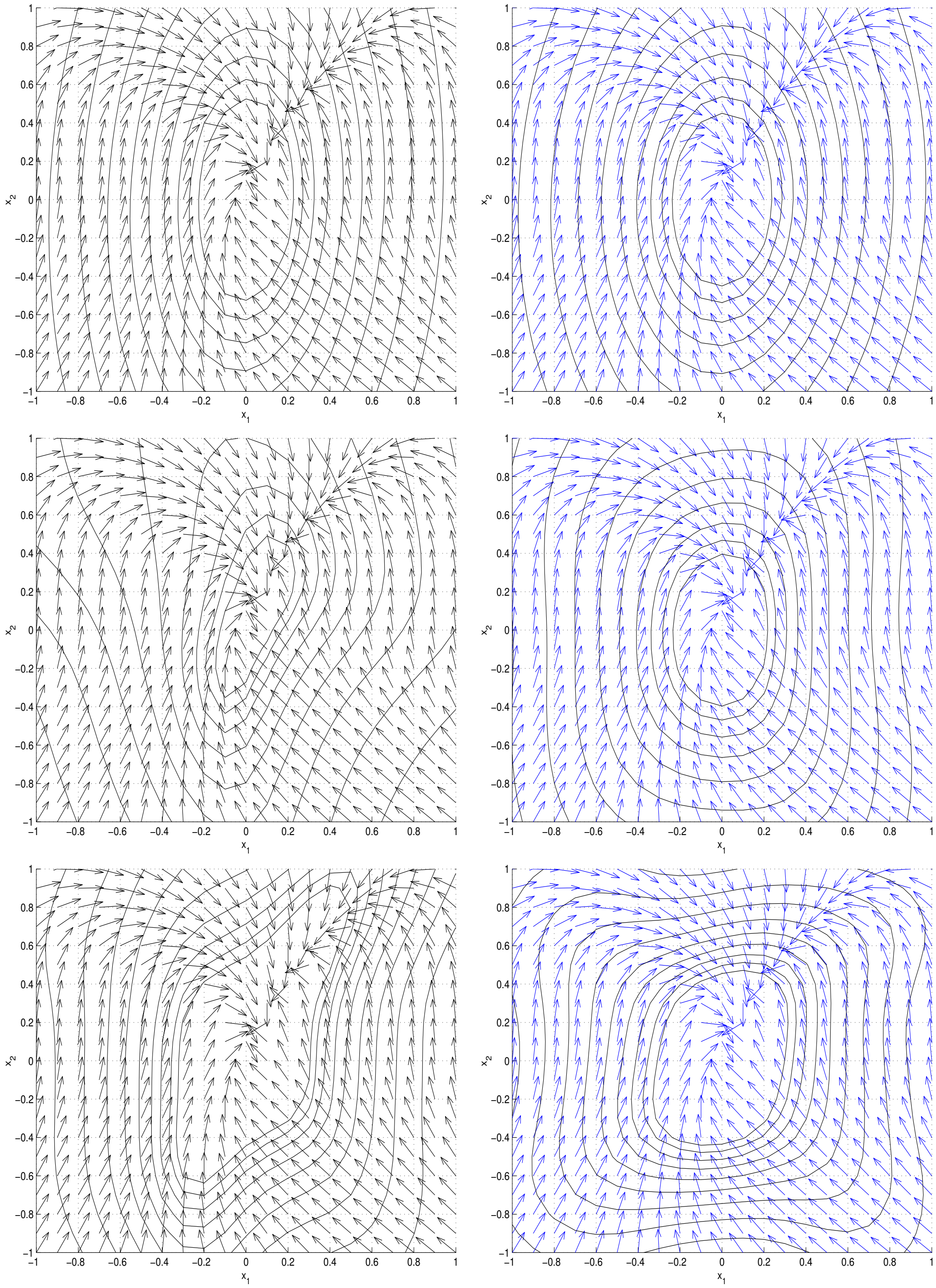


Figure 1: Computed Lyapunov functions for the autonomous system (22). The left column shows Lyapunov functions where the decay rate is maximized. The right column shows Lyapunov functions where the region of attraction is maximized. The first row are with parameterization $\mathcal{P}_1$, the second row with $\mathcal{P}_2$, and the third row with $\mathcal{P}_3$. Notice that the arrows only indicate the *direction* of the vector field $f(x)$ since they are normalized to have the same length (for the reason of improved presentation).

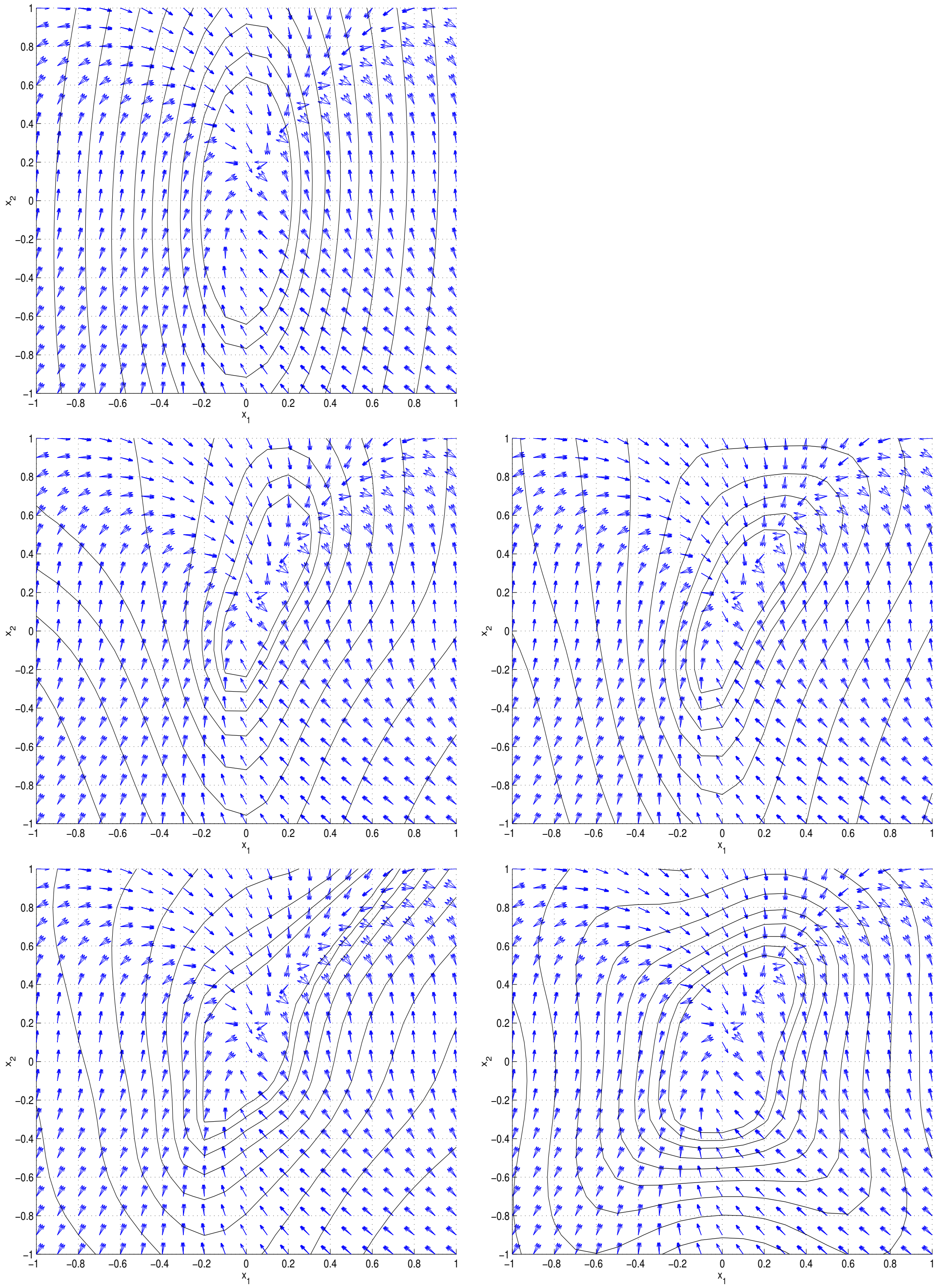

Figure 2: Computed Lyapunov functions for the non-autonomeous system (23). The left column shows Lyapunov functions where the decay rate is maximized. The right column shows Lyapunov functions where the region of attraction is maximized. The first row are with parameterization $\mathcal{P}_1$, the second row with $\mathcal{P}_2$, and the third row with $\mathcal{P}_3$. The arrows indicate the direction of the vector field $f(x,\theta)$ for $\theta \in \{-0.2, 0, 0.2\}$. Note that the vectors are normalized to have the same length (for the reason of improved presentation).