STABLE ADAPTIVE CONTROL OF A GENERAL CLASS OF NON-LINEAR SYSTEMS

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ABSTRACT

The properties of an adaptive control system based on a general class of non-linear models are analyzed. The class of models contains NARX models represented by feedforward neural networks with sigmoidal non-linearity, some radial basis-function expansions, local model networks, and some fuzzy systems. We apply a simple adaptive feedback linearizing controller. The analysis takes into account modeling error and slowly time-varying nominal model parameters. We discuss stability of the closed loop, as well as robustness, and derive performance bounds on the tracking error. Due to the general setup, the results are mainly of a qualitative nature. Finally, we discuss the relevance of the results, and outline some practical modifications and possible extensions.

1. Introduction

The analytical study of adaptive control loops involving complicated non-linear model structures and controllers such as neural networks and fuzzy systems has evolved considerably over the past five years. Some stability results for linearly parameterized model structures using feedback linearizing adaptive control structures include [56,55,46,25], while non-linearly parameterized control structures using feedback linearizing adaptive controllers are analyzed in [7,8,47,38,37,9].

Here we consider discrete-time, nonlinear systems whose input/output behavior can be adequately described by a predictor of the form

$$y(t+d) \quad = \quad f(y(t), ..., y(t-m), u(t), ..., u(t-r), \theta^\star(t)) + \nu(t+d) \qquad (1)$$

where f is a non-linear function. The input and output sequences u and y are assumed to be scalar, and m and r are non-negative integers. The integer $d-1 \geq 0$ is the system's time-delay. Alternatively, d can be interpreted as the system's relative degree [40]. The output of the function f is essentially the output predicted using the nominal model, so the sequence ν can be interpreted as unstructured uncertainty, which contains unmodeled dynamics (modeling error and reduced order modeling effects), disturbances, noise, effects due to sampling of continuous-time signals etc. The nominal parameter sequence $\theta^\star$ may be slowly time-varying.

The NARX-like model (nonlinear autoregressive with exogenous input) represented by (1) is quite general, although there may exist continuous- or discrete-time state-space models that do not have a global NARX representation of this form [36]. On the other hand, we introduce very weak assumptions on the structure of the function f. In particular, it can be represented by a wide range of series expansions, neural networks, fuzzy systems, wavelet expansions etc.

Compared to the other papers on control using non-linear model structures mentioned initially, the results here (see also [27]) rely on significantly weaker assumptions, which in particular allow slowly time-varying nominal parameters and a quite general form of unstructured model uncertainty. Weighted l_2-norms are applied in the analysis [14], which is the key to the strong qualitative results. Here we show that the global stability results in [27] quite naturally reduce to local stability results when the predictor is non-linearly parameterized. It is interesting to observe that the analysis based on weighted l_2-norms is quite analogous to the analysis of adaptive control systems based on discrete-time linear models [14]. There are also close links to the use of weighted L_2-norms when the model is continuous-time [60,21,30].

The continuation of this paper is as follows. First we review some basic results on weighted l_2-norms and state some general assumptions. The adaptive feedback linearizing controller is analyzed for the case of linear and non-linear parameterization of the predictor. Furthermore, the relevance to NARX model structures based on various neural and fuzzy model representations is studied. The assumptions are also closely examined. The paper ends with some practical considerations regarding some of the important design issues and concluding remarks.

2. Preliminaries

The Euclidean norm of a vector $x \in R^n$ is $||x|| = \sqrt{x^T x}$. The difference sequence Δs of a sequence $s = (s(t))$ is defined by $\Delta s(t) = s(t) - s(t-1)$ for all $t \geq 0$. The truncation of a sequence s at time t is denoted s_t. The exponentially weighted l_2-norm of a sequence is defined by

$$||s_t||_2^\delta = \sqrt{\sum_{\tau=0}^{t} \delta^{t-\tau} ||s(\tau)||^2}$$

for any $\delta \in [0,1]$ and $t \geq 0$. The norm of a discrete-time transfer function $H(q^{-1})$, where q^{-1} is the one-step delay operator and $H(q^{-1})$ has all poles strictly inside the unit disc, is defined by

$$||H(q^{-1})||_\infty = \sup_{\omega \in [0,2\pi]} |H\left(e^{-j\omega}\right)|$$

If $H(q^{-1})$ is proper and has all poles on or inside the disc $|c| \leq \sqrt{\delta}$, then the exponentially weighted norm is defined by $||H(q^{-1})||_\infty^\delta = ||H(\sqrt{\delta}q^{-1})||_\infty$. The mixed notation $s_2(t) = H(q^{-1})s_1(t)$, where $H(q^{-1})$ is an exponentially stable transfer function, means the convolution $s_2(t) = h(t) \star s_1(t)$, where $h(t)$ is the inverse z-transform of the transfer function $H(q^{-1})$. The exponentially decaying term, which may appear as a consequence of non-zero initial conditions, is consistently neglected in this work, as it does not affect the stability analysis. The following lemmas will be used frequently:

Lemma 1 *Let $s_2(t) = H(q^{-1})s_1(t)$, where $H(q^{-1})$ is a proper causal transfer function that has all poles in the disc $|c| \leq \sqrt{\delta}$, where $\delta \in (0,1]$, and suppose $\sum_{\tau=0}^{t} s_1^2(\tau) < \infty$. Then $||(s_2)_t||_2^\delta \leq ||H(q^{-1})||_\infty^\delta ||(s_1)_t||_2^\delta$ for all $t \geq 0$*

Proof: Given in [14].

□

Lemma 2 *Let s_1 be a positive sequence, i.e. $s_1(t) \geq 0$ for all t. Let the transfer function $H(q^{-1})$ be defined by $H(q^{-1}) = K(1-\sigma q^{-1})^{-1}$ for some $K \geq 0$ and $\sigma \in (0,1)$. Suppose the sequence s_2 satisfies*

$$s_2(t) \leq H(q^{-1})\left(\gamma_0 s_1(t) + \gamma_1 s_1(t-1) + ... + \gamma_n s_1(t-n)\right)$$

for some non-negative integer n and constants $\gamma_0, ..., \gamma_n \geq 0$. Then,

$$s_2(t) \leq \quad H(q^{-1})\left(\gamma_0 + \gamma_1 \sigma^{-1} + ... + \gamma_n \sigma^{-n}\right) s_1(t)$$

Proof:

$$s_2(t) \quad \leq \quad \sigma^t s_2(0) + K \sum_{\tau=0}^{t} \sigma^{t-\tau}\left(\gamma_0 s_1(\tau) + ... + \gamma_n s_1(\tau - n)\right) \tag{2}$$

Clearly, for any $k \in \{0, ..., n\}$

$$\begin{aligned}
\sum_{\tau=0}^{t} \sigma^{t-\tau}\gamma_k s_1(\tau-k) &= \sum_{\tau=-k}^{t-k} \sigma^{t-\tau-k}\gamma_k s_1(\tau) \\
&= \sigma^{-k}\sum_{\tau=0}^{t-k} \sigma^{t-\tau}\gamma_k s_1(\tau) + \text{exp. decaying term} \\
&\leq \sigma^{-k}\sum_{\tau=0}^{t} \sigma^{t-\tau}\gamma_k s_1(\tau) + \text{exp. decaying term}
\end{aligned}$$

where the last inequality follows because s_1 and γ_k are positive. Substituting into (2) gives the desired result.

$\square$

3. Problem Formulation and Basic Assumptions

Assumption AR. *Let $y^\star$ be a bounded reference sequence that is known d steps ahead in time. The bound is denoted $K^\star = \sup_{t \geq 0} |y^\star(t)|$.* $\square$

The problem we address here is the one of asymptotically tracking this reference, while rejecting noise and disturbances. In other words, if the tracking error is defined as $\tilde{y} = y^\star - y$, then the control objective is to make $|\tilde{y}(t)|$ as small as possible as $t \to \infty$.

Next, we will introduce some assumptions on the model and system, in terms of the model structure f, its sequence of nominal parameters $\theta^\star$, and the unstructured uncertainty ν. Thereafter, we will describe an on-line estimator for (4), as well as an adaptive control structure. The main result in this section concerns the stability, robustness, and performance of this adaptive controller.

Assumption AP. *There exists a constant $\underline{\theta} \geq 0$, such that $||\Delta\theta^\star(t)|| \leq \underline{\theta}$ for all $t \geq 0$. In addition, there exists a known convex and compact set Θ such that $\theta^\star(t) \in \Theta$ for all $t \geq 0$.* $\square$

This means that the parametric variation from time $t-1$ to t is bounded in norm by the small constant $\underline{\theta}$. As a consequence, there exists a $\overline{\theta} \geq 0$ such that $\theta_1, \theta_2 \in \Theta$ implies $||\theta_1 - \theta_2|| \leq \overline{\theta}$. This assumption is quite a standard one in the analysis of adaptive control loops, see e.g. [39,34].

Assumption AB. *The sign of the high-frequency gain is uniformly constant and known. Without loss of generality, we assume here that it is positive:*

$$\inf_{\theta \in \Theta} \inf_{y,u} \frac{\partial f(y(t), ..., y(t-m), u(t), ..., u(t-r), \theta)}{\partial u(t)} \;\geq\; \beta_0 \;>\; 0 \qquad (3)$$

It is assumed that f is a continuously differentiable function of $u(t)$. Moreover, f is bounded by an affine function, i.e. there exists a constant $C \geq 0$ such that

$$\sup_{\theta \in \Theta} ||f(y(t), ..., y(t-m), u(t), ..., u(t-r), \theta)|| \\ \leq C(1 + |y(t)| + ... + |y(t-m)| + |u(t)| + ... + |u(t-r)|)$$

$\square$

The first part of AB is essentially a strong global controllability assumption. In the usual global controllability definition, it is required that there exists an admissible control input such that any state can be reached from any initial state within an unspecified, but finite time [43]. Our definition requires this time to be d, which is the shortest possible. On the other hand, we are only concerned about the output, not the full state.

The second part of AB is made necessary by the use of input/output stability theory, and is essentially the reason why the same tools that are used to study adaptive control on the basis of linear models can be used to study adaptive control using quite general nonlinear models of the form (1). This growth condition is not too restrictive, since all systems and models where the inputs and outputs do not grow or decay at a rate that is faster than exponential will satisfy this assumption.

4. Linearly Parameterized Model Representation

To begin with, we assume the predictor (1) is linearly parameterized:

$$y(t+d) \quad = \quad \varphi^T(y(t),...,y(t-m),u(t),...,u(t-r))\theta^\star(t)+\nu(t+d) \tag{4}$$

where the definition of φ follows directly from the identity $f(\cdot,\theta)=\varphi^T(\cdot)\theta$, cf. (1).

4.1. Adaptive Control Structure

The fundamentals of discrete-time feedback linearization are studied in [40]. The idea is to render the non-linear nominal system linear with the aid of a static feedback. In general, this is not always possible, as one should expect, but with the above model representation, the adaptive certainty equivalence feedback linearizing controller is well defined by the implicit equation

$$v(t) \quad = \quad \varphi^T(y(t),...,y(t-m),u(t),...,u(t-r))\hat{\theta}(t) \tag{5}$$

where $\hat{\theta}(t)$ is an estimate of $\theta^\star(t)$. From (3) it follows that for each $v(t)$ there always exists a unique control input $u(t)$ that satisfies (5). The closed loop behavior is now described by

$$y(t) \quad = \quad q^{-d}v(t)+\varphi^T(t-d)(\theta^\star(t-d)-\hat{\theta}(t-d))+\nu(t) \tag{6}$$

The nominal behavior from v to y is simply the linear system with transfer function q^{-d}, because of the feedback linearization. The second term is the parametric error due to the fact that the controller only has estimates of the nominal parameters. Finally, the third term is the unstructured uncertainty. The following linear control algorithm is applied to control the above nominally linear system

$$v(t) \quad = \quad y^\star(t+d)+G(q^{-1})(y^\star(t)-y(t)) \tag{7}$$

where $G(q^{-1}) = P(q^{-1})/Q(q^{-1})$. Combining (6) and (7), we see that the closed loop system satisfies

$$\tilde{y}(t) \;=\; M(q^{-1})\left(\varphi^T(t-d)(\hat{\theta}(t-d) - \theta^\star(t-d)) - \nu(t)\right) \qquad (8)$$

where

$$M(q^{-1}) \;=\; \frac{1}{1+G(q^{-1})q^{-d}} \;=\; \frac{Q(q^{-1})}{Q(q^{-1}) + q^{-d}P(q^{-1})}$$

is by design a stable transfer function with a bandwidth that equals the desired bandwidth of the closed loop. The effect of the parametric and unstructured uncertainty in (8) will be analyzed in the remaining of this section. Notice that this is a non-trivial task because boundedness of φ and ν are not established a priori.

4.2. Parameter Estimation

The estimate $\hat{\theta}(t)$ of $\theta^\star(t)$ is based on the predictor $\hat{y}(t|t-1) = \varphi^T(t-d)\hat{\theta}(t-1)$ which gives the prediction error

$$\begin{aligned} e(t) &= \hat{y}(t|t-1) - y(t) \\ &= \varphi^T(t-d)\hat{\theta}(t-1) - \varphi^T(t-d)\theta^\star(t-d) - \nu(t) \end{aligned} \qquad (9)$$

and normalized prediction error $\varepsilon(t) = e(t)/n(t)$, where the scalar normalizing sequence n is defined by

$$n^2(t) \;=\; \delta n^2(t-1) + ||\varphi(t-d)||^2 + 1$$

with arbitrary $n(0) > 0$ and $\delta \in (0,1)$ given. The normalization is introduced to justify that the normalized unstructured uncertainty is bounded [49]:

Assumption AU. *The unstructured uncertainty satisfies* $|\nu(t)| \leq \mathcal{V}n(t)$ *for some known constant* $\mathcal{V} \geq 0$. $\square$

The normalized error signal is a standard tool for dealing with unstructured uncertainty. This formulation is widely applied in the field of adaptive control, both in a continuous and discrete time framework, e.g. [21]. As we shall see later, it is straightforward to show that this assumption allows a quite general class of additive and multiplicative modeling error due to, for instance, neglected higher-order dynamics or structural mismatch of the model non-linearities, see also [35].

To estimate the unknown nominal parameters, we apply a standard recursive parameter estimation algorithm with normalization and relative dead-zone [35]:

$$\hat{\theta}'(t) \;=\; \hat{\theta}(t-1) - \lambda\frac{\varphi(t-d)}{1+\varphi^T(t-d)\varphi(t-d)}n(t)\mathcal{D}\left(\varepsilon(t)\right) \qquad (10)$$

$$
\hat{\theta}(t) = \mathcal{P}_{\Theta}\left(\hat{\theta}'(t)\right) \tag{11}
$$

$$
\mathcal{D}(\varepsilon(t)) = \begin{cases} \varepsilon(t)+d_0 & \text{if } \varepsilon(t) < -d_0 \\ 0 & \text{if } |\varepsilon(t)| \leq d_0 \\ \varepsilon(t)-d_0 & \text{if } \varepsilon(t) > d_0 \end{cases} \tag{12}
$$

where $\mathcal{P}_{\Theta}$ is a continuous parameter projection that projects its argument to the closest point (using Euclidean norm) in Θ. The continuous function $\mathcal{D}$ is referred to as a dead-zone function, the constant $d_0 \geq 0$ is the magnitude of the dead-zone, and $\lambda > 0$ is the estimator gain. For convenience, the parameter estimation error sequence is defined by $\tilde{\theta}(t) = \hat{\theta}(t) - \theta^{\star}(t)$. Next, we examine the properties of this algorithm:

Theorem 1 *Suppose the outputs are generated by*

$$
y(t+d) = \varphi^T(t)\theta^{\star}(t) + \nu(t+d)
$$

and consider the parameter estimation algorithm (10)-(12) with initial estimate $\hat{\theta}(0) \in \Theta$. *If AP and AU hold, the estimator gain satisfies* $\lambda \in (0,2)$, *and the dead-zone is chosen such that* $d_0 = \mathcal{V}$, *then the algorithm has the properties*

$$
\begin{aligned}
\sum_{\tau=0}^{t} \mathcal{D}^2(\varepsilon(\tau)) &\leq \left(\frac{2}{\lambda(2-\lambda)}\left((d\underline{\theta})^2 + 2(1+\lambda)d\underline{\theta}\overline{\theta}\right) + 2(d-1)^2\underline{\theta}^2\right)t \\
&\quad + \frac{2}{\lambda(2-\lambda)}\overline{\theta}^2
\end{aligned} \tag{13}
$$

$$
|\varepsilon(t)| \leq \mathcal{V} + |\mathcal{D}(\varepsilon(t))| \tag{14}
$$

$$
\begin{aligned}
\sum_{\tau=0}^{t} ||\Delta\hat{\theta}(\tau)||^2 &\leq \left(\frac{\lambda}{2-\lambda}\left((d\underline{\theta})^2 + 2(1+\lambda)d\underline{\theta}\overline{\theta}\right) + (d-1)^2\underline{\theta}^2 + 2\lambda(d-1)\underline{\theta}\overline{\theta}\right)t \\
&\quad + \frac{\lambda}{2-\lambda}\overline{\theta}^2
\end{aligned} \tag{15}
$$

for all $t \geq 0$.

Proof: Rewriting the equation for the prediction error (9) gives

$$
\varepsilon(t) = \frac{\varphi^T(t-d)\tilde{\theta}(t-1)}{n(t)} - \frac{\varphi^T(t-d)}{n(t)}\left(\theta^{\star}(t-d) - \theta^{\star}(t-1)\right) - \frac{\nu(t)}{n(t)}
$$

Since $d_0 = \mathcal{V}$ there exists a sequence β with the properties $0 \leq \beta(t) \leq 1$ and

$$
\mathcal{D}(\varepsilon(t)) = \beta(t)\frac{\varphi^T(t-d)\tilde{\theta}(t-1)}{n(t)} - \beta(t)\frac{\varphi^T(t-d)}{n(t)}\left(\theta^{\star}(t-d) - \theta^{\star}(t-1)\right)
$$

From (10) we find

$$
\tilde{\theta}'(t) = \left(I - \lambda\beta(t)\frac{\varphi(t-d)\varphi^T(t-d)}{1+\varphi^T(t-d)\varphi(t-d)}\right)\tilde{\theta}(t-1) + \gamma(t) \tag{16}
$$

where

$$\gamma(t) = -\Delta\theta^\star(t) + \lambda\beta(t)\frac{\varphi(t-d)\varphi^T(t-d)}{1+\varphi^T(t-d)\varphi(t-d)}\left(\theta^\star(t-d) - \theta^\star(t-1)\right)$$

Defining the function $V(t) = ||\tilde{\theta}(t)||^2$, we find $V(t) \leq ||\tilde{\theta}'(t)||^2$ from the convexity of Θ. Hence,

$$\begin{aligned} V(t) - V(t-1) \leq & \left(\gamma^T(t) + \tilde{\theta}^T(t-1)\left(I - \lambda\beta(t)\frac{\varphi(t-d)\varphi^T(t-d)}{1+\varphi^T(t-d)\varphi(t-d)}\right)\right) \\ & \cdot \left(\left(I - \lambda\beta(t)\frac{\varphi(t-d)\varphi^T(t-d)}{1+\varphi^T(t-d)\varphi(t-d)}\right)\tilde{\theta}(t-1) + \gamma(t)\right) \\ & -\tilde{\theta}^T(t-1)\tilde{\theta}(t-1) \end{aligned}$$

Assume $\beta(t) \neq 0$, then

$$\begin{aligned} V(t) - V(t-1) \leq \; & \gamma^T(t)\gamma(t) \\ & +2\gamma^T(t)\left(I - \lambda\beta(t)\frac{\varphi(t-d)\varphi^T(t-d)}{1+\varphi^T(t-d)\varphi(t-d)}\right)\tilde{\theta}(t-1) \\ & -\left(\frac{2\lambda}{\beta(t)} - \lambda^2\right)\left(\frac{n^2(t)}{1+\varphi^T(t-d)\varphi(t-d)}\right) \\ & \cdot\left(\mathcal{D}(\varepsilon(t)) + \beta(t)\frac{\varphi^T(t-d)}{n(t)}\left(\theta^\star(t-d) - \theta^\star(t-1)\right)\right)^2 \end{aligned}$$

or

$$\begin{aligned} & \left(\mathcal{D}(\varepsilon(t)) + \beta(t)\frac{\varphi^T(t-d)}{n(t)}\left(\theta^\star(t-d) - \theta^\star(t-1)\right)\right)^2 \leq \\ & \qquad \frac{\beta(t)}{\lambda(2-\beta(t)\lambda)}\left(||\gamma(t)||^2 + 2(1+\lambda)||\gamma(t)|| \cdot ||\tilde{\theta}(t-1)|| - \Delta V(t)\right) \end{aligned} \tag{17}$$

If $\beta(t) = 0$, then $V(t) = V(t-1)$ and $\mathcal{D}(\varepsilon(t)) = 0$. Hence, from the properties of β it follows that

$$\begin{aligned} \mathcal{D}^2(\varepsilon(t)) \leq \; & \frac{2}{\lambda(2-\lambda)}\left(||\gamma(t)||^2 + 2(1+\lambda)||\gamma(t)|| \cdot ||\tilde{\theta}(t-1)|| - \Delta V(t)\right) \\ & +2||\theta^\star(t-d) - \theta^\star(t-1)||^2 \end{aligned}$$

Summing, and using the fact that $||\tilde{\theta}(t)|| \leq \overline{\theta}$, which implies $V(t) \leq \overline{\theta}^2$ for all $t \geq 0$, together with the fact that AP implies $||\gamma(t)|| \leq d\underline{\theta}$, we get (13). By the definition of the dead-zone function, (14) follows. Finally, we get from (16) that

$$
\begin{aligned}
\sum_{\tau=0}^{t} ||\Delta\hat{\theta}(\tau)||^2 &\leq \sum_{\tau=0}^{t} ||\hat{\theta}'(\tau) - \hat{\theta}(\tau-1)||^2 \\
&\leq \sum_{\tau=0}^{t} \Big((\gamma(\tau) + \Delta\theta^\star(\tau))^T (\gamma(\tau) + \Delta\theta^\star(\tau)) \\
&\quad -2(\gamma(\tau) + \Delta\theta^\star(\tau))^T \lambda\beta(\tau) \frac{\varphi(\tau-d)\varphi^T(\tau-d)}{1+\varphi^T(\tau-d)\varphi(\tau-d)} \tilde{\theta}(\tau-1) \\
&\quad + \lambda^2\beta^2(\tau)\tilde{\theta}^T(\tau-1) \frac{\varphi(\tau-d)\varphi^T(\tau-d)\varphi(\tau-d)\varphi^T(\tau-d)}{(1+\varphi^T(\tau-d)\varphi(\tau-d))^2} \tilde{\theta}(\tau-1) \Big) \\
&\leq \sum_{\tau=0}^{t} \Big(||\gamma(\tau) + \Delta\theta^\star(\tau)||^2 + 2\lambda ||\gamma(\tau) + \Delta\theta^\star(\tau)|| \cdot ||\tilde{\theta}(\tau-1)|| \\
&\qquad + \lambda^2 \frac{n^2(\tau)}{1+\varphi^T(\tau-d)\varphi(\tau-d)} \\
&\qquad \Big(\mathcal{D}(\varepsilon(\tau)) + \beta(\tau) \frac{\varphi^T(\tau-d)}{n(\tau)} (\theta^\star(\tau-d) - \theta^\star(\tau-1)) \Big)^2 \Big)
\end{aligned}
$$

Now, (15) follows from (17).

□

The relative dead-zone is a modification that turns off the adaptation of the parameter estimate when the normalized prediction error becomes "small", therefore preventing drift phenomena that otherwise might be excited by the unstructured uncertainty. There are, however, several other modifications available that lead to parameter estimation algorithms with similar properties, including different variations of the σ-modification, ϵ-modification, and parameter projection [42,21,14,64]. The design of adaptive schemes that retain their closed loop stability properties in the presence of not only large parametric uncertainty but also modeling errors (such as additive disturbances and unmodelled dynamics) is referred to as *robust adaptive control*; a unified treatment of this area can be found in [21].

4.3. Closed Loop Stability

It is well known that input-output feedback linearizing control structures may render some of the system states unobservable [40,6]. In order to ensure boundedness of these states, we must study the behavior of the inverse system. By the Implicit Function Theorem [43], and assumption AB, it is evident that the inverse system is globally defined by a function g:

$$u(t) = g(u(t-1), ..., u(t-r), y(t+d), y(t), ..., y(t-m), \nu(t+d), \theta^\star(t)) \quad (18)$$

where y and ν are viewed as inputs, and u as the output.

Assumption AI. *The inverse nominal model (18) is globally uniformly exponentially stable, in the sense that for arbitrary initial conditions, there exists constants K_σ and $\sigma \in [0,1)$ such that the impulse response coefficients of (18) are bounded by an exponentially decaying sequence $(K_\sigma \sqrt{\sigma}^t)$.* □

Stability of the inverse nominal model is often referred to as a minimum phase property of the model [5]. It should be stressed that with the model representation and control structure outlined above, the system can actually be non-minimum phase, for the following two reasons:

1. The particularly simple form of the model has allowed us to factor out the time delay $q^{-(d-1)}$ from the nominal model, which is clearly a non-minimum phase phenomenon. Notice that this is in contrast to the continuous-time case where the representation of time-delays is less straightforward. Also notice that the factorization of non-linear systems is in general a complicated and very much unsolved problem [20,15,2]. This is in sharp contrast to the linear case, where non-minimum phase phenomena can easily be factored out and identified as zeros outside the unit circle in the complex plane.

2. There may be non-minimum phase phenomena hidden in the unstructured uncertainty. As we shall see, stability of the adaptive control system requires the unstructured uncertainty to be small, so these non-minimum phase phenomena should not be too large. In practise, we expect that non-minimum phase phenomena due to fast sampling of a continuous-time system [1,11] are small enough to be regarded as unmodeled dynamics.

Under Assumption AI, we can prove the following lemma, which essentially shows that the input is bounded by the output and the unstructured uncertainty:

Lemma 3 *Suppose Assumptions AB and AI hold, then for any $\kappa \in (\sigma, 1)$*

$$||(q^{-d}u)_t||_2^\kappa \;\leq\; \frac{K_\sigma}{1-\sqrt{\sigma/\kappa}} \left(||y_t||_2^\kappa + ||\nu_t||_2^\kappa\right)$$

Proof. It follows directly from Assumption AI that

$$|u(t)| \;\leq\; K_\sigma \sum_{\tau=0}^{t} \sqrt{\sigma}^{t-\tau} \left(|y(\tau+d)| + |\nu(\tau+d)|\right)$$

The result follows from Lemma 1 together with

$$||(1-\sqrt{\sigma}q^{-1})^{-1}||_\infty^\kappa \;=\; (1-\sqrt{\sigma/\kappa})^{-1}$$

□

We are now in position to prove the main result:

Theorem 2 *Suppose the system (4) is controlled by (5) and (7), and the algorithm (10)-(12) is applied to estimate the parameter sequence $\theta^\star$. Suppose Assumptions AB, AR, AP, AU and AI hold, and let $\rho \in (0,1)$ and K_ρ be constants such that the impulse response coefficients of $M(q^{-1})$ are bounded by $K_\rho \sqrt{\rho}^t$. Suppose the magnitude of the dead-zone is chosen as $d_0 = \mathcal{V}$. Let κ be arbitrary in the open interval $\max(\delta, \sigma, \rho) < \kappa < 1$. If $\underline{\theta}$ and $\mathcal{V}$ are both sufficiently small, then for arbitrary initial conditions and $\hat{\theta}(0) \in \Theta$, the input and output sequences u and y are bounded, and the tracking error satisfies*

$$\frac{1}{t}\sum_{\tau=0}^{t} \tilde{y}^2(\tau) \quad \leq \quad c_1 + \frac{1}{t} c_2 \tag{19}$$

for all $t > 0$, where c_2 is proportional to $\overline{\theta}^2$, and $c_1 \to 0$ when $\underline{\theta} \to 0$ and $\mathcal{V} \to 0$.

Proof: Let us define

$$\tilde{Y}(t) \quad = \quad \varphi^T(t-d)\tilde{\theta}(t-d) - \nu(t) \tag{20}$$

Combining (9) and (20), we get

$$\frac{\tilde{Y}(t)}{n(t)} \quad = \quad \varepsilon(t) + \frac{\varphi^T(t-d)}{n(t)} \left(\hat{\theta}(t-d) - \hat{\theta}(t-1)\right) \tag{21}$$

From the definition of the normalizing signal, assumption AB and Lemma 2, it is evident that there exists a constant C such that

$$n^2(t) \quad \leq \quad \frac{1}{1-\delta} + C^2 \sum_{\tau=0}^{t} \kappa^{t-\tau} \left(1 + |y(\tau-d)| + |u(\tau-d)|\right)^2$$

Using Lemma 3 and the triangle inequality, we find

$$n^2(t) \quad \leq \quad \frac{1+3C^2}{1-\kappa} + 3C^2 \left(\left|\left| \left(q^{-d} y\right)_t \right|\right|_2^\kappa \right)^2 + \frac{3C^2 K_\sigma^2}{(1-\sqrt{\sigma/\kappa})^2} \left(||(|y| + |\nu|)_t||_2^\kappa \right)^2$$

because $\kappa > \sigma$. Using Lemma 2, we get

$$n^2(t) \quad \leq \quad \frac{1+3C^2}{1-\kappa} + (K_y\, ||y_t||_2^\kappa)^2 + (K_\nu\, ||\nu_t||_2^\kappa)^2$$

where $K_y = \sqrt{3}C(\kappa^{-d/2} + \sqrt{2}K_\sigma(1-\sqrt{\sigma/\kappa})^{-1})$ and $K_\nu = \sqrt{6}CK_\sigma(1-\sqrt{\sigma/\kappa})^{-1}$. By the triangle inequality, the bound on $y^\star$, and (8), we get

$$n^2(t) \quad \leq \quad K_0 + \left(K_y (1-\sqrt{\rho/\kappa})^{-1} ||\tilde{Y}_t||_2^\kappa \right)^2 + (K_\nu ||\nu_t||_2^\kappa)^2$$

where $K_0 = (1 + 3C^2 + 3C^2 K_y^2 (K^\star)^2)/(1-\kappa)$. This can be written

$$n^2(t) \quad \leq \quad K_0 + K_y^2 \frac{\kappa}{(\sqrt{\kappa}-\sqrt{\rho})^2} \sum_{\tau=0}^{t} \kappa^{t-\tau} \tilde{Y}^2(\tau) + K_\nu^2 \sum_{\tau=0}^{t} \kappa^{t-\tau} \nu^2(\tau)$$

After some straightforward manipulations and using (21)

$$n^2(t) \quad \leq \quad K_0 + K_y^2 \frac{\kappa}{(\sqrt{\kappa}-\sqrt{\rho})^2} \sum_{\tau=0}^{t} \kappa^{t-\tau} s^2(\tau) n^2(\tau) + K_\nu^2 \sum_{\tau=0}^{t} \kappa^{t-\tau} \nu^2(\tau)$$

where we have defined

$$s^2(t) \quad = \quad \left(\varepsilon(t) + \frac{\varphi^T(t-d)}{n(t)} \left(\hat{\theta}(t-d) - \hat{\theta}(t-1) \right) \right)^2 \tag{22}$$

It is straightforward to show that there exists a finite constant $B \geq 1$ such that

$$\frac{n^2(t+1)}{n^2(t)} \quad \leq \quad B$$

for all $t \geq 0$. Hence

$$n^2(t) \quad \leq \quad K_0 + \sum_{\tau=0}^{t-1} \kappa^{t-\tau} w^2(\tau) n^2(\tau) \tag{23}$$

where $w^2(t) = B\kappa^{-1} K_\nu^2 \mathcal{V}^2 + B K_y^2 (\sqrt{\kappa}-\sqrt{\rho})^{-2} s^2(t)$. By the Gronwall lemma, e.g. [16,14], and the relation between arithmetic and geometric means, e.g. [53], we get from (23)

$$n^2(t) \quad \leq \quad K_0 + K_0 \sum_{\tau=0}^{t-1} \kappa^{t-\tau} w^2(\tau) \left(1 + \frac{\kappa^{-1}}{t-\tau-1} \sum_{i=\tau+1}^{t-1} w^2(i) \right)^{t-\tau-1}$$

From Lemma 1, it follows after some manipulation that

$$\sum_{\tau=0}^{t} w^2(\tau) \quad \leq \quad \mu_0 t + \mu_1 \tag{24}$$

where

$$\begin{aligned} \mu_0 \quad &= \quad \left(B\kappa^{-1} K_\nu^2 + \frac{2BK_y^2}{(\sqrt{\kappa}-\sqrt{\rho})^2} \right) \mathcal{V}^2 + \frac{4Bd^2 K_y^2}{(\sqrt{\kappa}-\sqrt{\rho})^2} \left(\frac{3+\lambda}{\lambda(2-\lambda)} \right) \underline{\theta}^2 \\ &\quad + \frac{2BdK_y^2}{(\sqrt{\kappa}-\sqrt{\rho})^2} \left(\frac{\lambda^3 + 4\lambda^2 + 4\lambda + 4}{\lambda(2-\lambda)} \right) \overline{\theta}\underline{\theta} \qquad (25) \\ \mu_1 \quad &= \quad \frac{2BK_y^2}{(\sqrt{\kappa}-\sqrt{\rho})^2} \left(\frac{\lambda^2+2}{\lambda(2-\lambda)} \right) \overline{\theta}^2 \end{aligned}$$

Now,

$$n^2(t) \le K_0 + K_0 \sum_{\tau=0}^{t-1} \kappa^{t-\tau} w^2(\tau)(1+\kappa^{-1}\mu_0)^{t-\tau-1} \left(1 + \frac{\mu_1}{(\kappa+\mu_0)(t-\tau-1)}\right)^{t-\tau-1}$$

Since $(1+x^{-1})^x \le e$ for all $x > 0$, it follows that

$$n^2(t) \;\le\; K_0 + K_0(1+\kappa^{-1}\mu_0)^{-1} \exp\left(\frac{\mu_1}{\kappa+\mu_0}\right) \sum_{\tau=0}^{t-1} (\kappa+\mu_0)^{t-\tau} w^2(\tau)$$

Taking $\underline{\theta}$ and $\mathcal{V}$ sufficiently small, it follows from (25) that $\mu_0 + \kappa < 1$. Hence, (cf. Lemma 2.4 in [14])

$$n^2(t) \;\le\; K_0 + \frac{K_0(\mu_0+\mu_1)\exp\left(\frac{\mu_1}{\kappa+\mu_0}\right)}{(1-\kappa-\mu_0)(1+\kappa^{-1}\mu_0)} \;=\; \overline{n}^2$$

and it is proved that n is a bounded sequence. From (8) and Lemma 1 it follows that

$$\begin{aligned} \sum_{\tau=0}^{t} \tilde{y}^2(\tau) \;&\le\; \frac{1}{(1-\sqrt{\rho})^2} \sum_{\tau=0}^{t} \tilde{Y}^2(\tau) \;\le\; \frac{1}{(1-\sqrt{\rho})^2} \sum_{\tau=0}^{t} s^2(\tau) n^2(\tau) \\ &\le\; \frac{2\overline{n}^2}{(1-\sqrt{\rho})^2} \sum_{\tau=0}^{t} \left(\varepsilon^2(\tau) + (d-1)^2 \left\|\Delta\hat{\theta}(\tau)\right\|^2\right) \end{aligned}$$

From (22) and Lemma 1, it is evident that (19) holds, and c_1 and c_2 have the stated properties. It is easy to see that $\tilde{y}$ must be a bounded sequence. Hence, y is bounded since $y^\star$ is bounded. Since ν is bounded by n, it follows from Lemma 3 that u is bounded as well.

□

Theorem 2 provides sufficient conditions for robust stability of the adaptive control loop. Stability requires the bounds $\mathcal{V}$ and $\underline{\theta}$ on the unstructured uncertainty and nominal parameter variations both to be sufficiently small. The degree of robustness can the quantified in terms of the inequality $\mu_0 + \kappa < 1$ since $\kappa < 1$ is a given constant and μ_0 depends on $\mathcal{V}$ and $\underline{\theta}$. Hence, μ_0 can be interpreted as a sort of stability margin. However, both the stability margin and the performance result (19) can potentially be very conservative, and are best interpreted in a qualitative way.

We see from (19) that the average squared tracking error is bounded by a sum of two terms. The first term is the asymptotic performance bound that scales with the uncertainty and parameter variations. The second term is a bound on the transient performance that scales with the bound $\overline{\theta}^2$ on the parameter set, and vanishes asymptotically at the rate $1/t$. Hence, the average squared tracking error will tend towards a neighborhood of the origin at the rate $1/t$. Unfortunately, we cannot provide any strong results about the instantaneous value of the tracking error, except that it is bounded. Various phenomena like bursting may therefore be present [64].

5. Non-linearly Parameterized Model Structure

In this section we extend the results of the previous section to models of the form (1). In other words, the predictor is allowed to be nonlinearly parameterized.

Suppose the certainty equivalence adaptive feedback linearizing controller

$$v(t) = f(y(t), ..., y(t-m), u(t), ..., u(t-r), \hat{\theta}(t)) \tag{26}$$

is applied. Under smoothness assumptions on f, Taylor's theorem immediately gives the following error equation [48]

$$\begin{aligned} \tilde{y}(t) &= M(q^{-1}) \left(\frac{\partial f}{\partial \theta}(\cdot, \hat{\theta}(t-d))\tilde{\theta}(t-d) \right. \\ &\qquad \left. + \frac{1}{2}\tilde{\theta}(t-d)\frac{\partial^2 f}{\partial \theta^2}(\cdot, \overline{\theta}(t-d))\tilde{\theta}(t-d) - \nu(t) \right) \end{aligned}$$

where $\overline{\theta}(t) = \alpha(t)\theta^\star(t) + (1-\alpha(t))\hat{\theta}(t)$ for some $\alpha(t) \in [0,1]$. Defining

$$\begin{aligned} \varphi(t) &= \frac{\partial f}{\partial \theta}(\cdot, \hat{\theta}(t)) \\ \eta(t) &= \nu(t) - \frac{1}{2}\tilde{\theta}(t-d)\frac{\partial^2 f}{\partial \theta^2}(\cdot, \overline{\theta}(t-d))\tilde{\theta}(t-d) \end{aligned}$$

we get

$$\tilde{y}(t) = M(q^{-1})(\varphi^T(t-d)\tilde{\theta}(t-d) - \eta(t))$$

By substituting the unstructured uncertainty $\nu(t)$ by $\eta(t)$, which also includes higher-order terms (with respect to the parameter error $\tilde{\theta}$), it is now straightforward to extend the results of Theorem 2 to nonlinearly parameterized systems.

Theorem 3 *Suppose the system (1) is controlled by (26) and (7), and the algorithm (10)-(12) is applied to estimate the parameter sequence $\theta^\star$. Suppose all the assumptions of Theorem 2 hold, and in addition*

1. *The function f is twice continuously differentiable with respect to θ, there exists a constant $K \geq 0$ such that its Hessian satisfies*

$$\sup_{y,u} \sup_{\theta \in \Theta} \left|\left| \frac{\partial f^2}{\partial \theta^2}(y(t), ..., y(t-m), u(t), ..., u(t-r), \theta) \right|\right|_2 \leq K$$

   where $||\cdot||_2$ is the induced matrix norm.

2. *The bound on the parameter set $\overline{\theta}$ is sufficiently small.*

3. *The magnitude of the dead-zone is chosen as $d_0 = \mathcal{V} + K\overline{\theta}^2/2$.*

Then the conclusion of Theorem 2 still holds.

Proof: It is clear that $\eta(t) \leq \mathcal{V}' n(t)$ where

$$\mathcal{V}' = \mathcal{V} + \frac{K}{2}\overline{\theta}^2 = d_0$$

Hence, since $\underline{\theta}, \overline{\theta}$ and $\mathcal{V}$ are all sufficiently small, then Lemma 1 and Theorem 2 hold.

□

As one should expect, the global stability result in Theorem 2 reduces to a local stability result when the predictor is not linearly parameterized. Hence, the main price to pay for the relaxed assumption is that the bound on the parametric uncertainty $\overline{\theta}$ must be sufficiently small. In addition, it was necessary to introduce a regularity assumption on f and to increase the size of the dead-zone to account for the higher-order effects. Since Assumption AB requires f to be uniformly bounded by an affine function, it is clear that the uniform boundedness of the Hessian is a quite reasonable assumption. For instance, if f is Lipschitz, it will satisfy the regularity condition. Unfortunately, the increased dead-zone will potentially give even more conservative performance bounds than in the linear parameterization case.

6. Nominal Model Uncertainty, Uniform Approximation, Neural Networks and other Model Structures

The choice of model structure is of major importance in model-based control. In this section we investigate to what extent certain generic non-linear model structures satisfy the assumptions made in the analysis. We concentrate on Assumptions AB, AP and AU.

6.1. Model Uncertainty

A major difference between Assumption AB on one side, and Assumptions AP and AU on the other side, is that assumption AB concerns the model structure and parameter set, which are subject to design, while assumptions AP and AU are concerned with the relationship between the nominal model and the underlying system, and are not only difficult to verify, but are also outside the influence of the control law designer. In general, assumptions AP and AU require the nominal model to be capable of closely approximating the input/output behavior of the system. It may therefore be desirable to choose the function f such that the resulting model structure is a close approximation to a large class of non-linear systems.

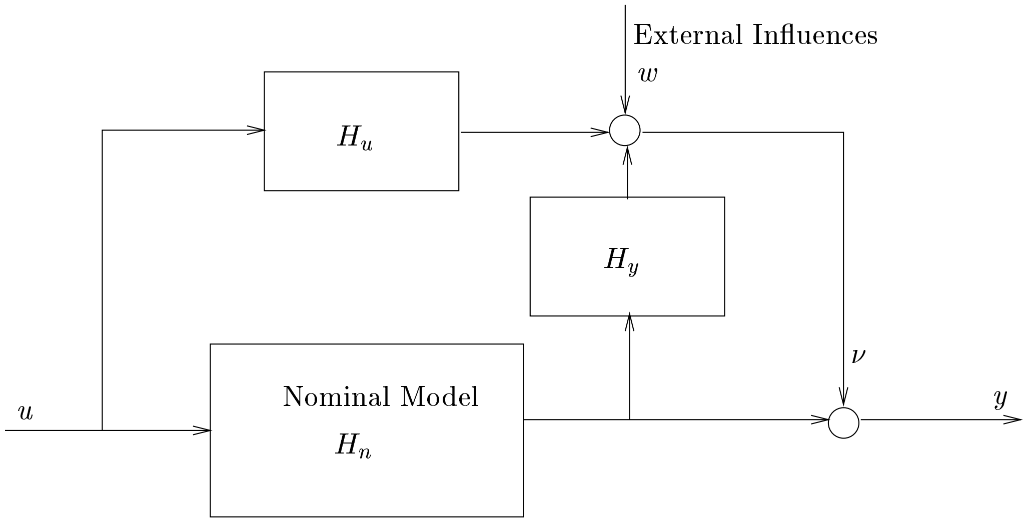

Figure 1: Nominal model and uncertainty.

Next, we proceed with an example of unstructured uncertainty that satisfies assumption AU. Assume the nominal model defined in (1) can be described as a possibly nonlinear operator H_n, and we have additive unmodeled dynamics H_u and multiplicative unmodeled dynamics H_y in addition to external disturbances and noise w as shown in Figure 1:

$$y = H_n u + \nu \tag{27}$$

$$\nu = H_u u + H_y H_n u + w \tag{28}$$

which gives

$$\begin{aligned} y &= H_n u + H_u u + H_y H_n u + w \\ &= (I + H_y) H_n u + H_u u + w \end{aligned}$$

where the three terms clearly contain multiplicative uncertainty, additive uncertainty, and external influences, respectively.

Lemma 4 *Suppose Assumption AB holds, and in addition*

$$
\begin{aligned}
H_y(q^{-1}) &= \varepsilon_y \sum_{i=0}^{\infty} \alpha_i q^{-i}, \quad \sum_{i=0}^{\infty} |\alpha_i| \sqrt{\delta}^{-i} < \infty \\
H_u(q^{-1}) &= \varepsilon_u \sum_{i=0}^{\infty} \beta_i q^{-i}, \quad \sum_{i=0}^{\infty} |\beta_i| \sqrt{\delta}^{-i} < \infty \\
|w(t)| &\leq \varepsilon_w n(t)
\end{aligned}
$$

Then $\nu(t) \leq \mathcal{V} n(t)$ *where* $\mathcal{V} = K \max(\varepsilon_y, \varepsilon_u, \varepsilon_w)$ *and* K *is a constant.*

Proof. It is straightforward to see that there exist a constant K_y such that

$$
\begin{aligned}
|(H_n u)(t)| &= |f(y(t-d), ..., y(t-d-m), u(t-d), ..., u(t-d-r), \theta^\star(t))| \\
&\leq C\sqrt{1 + ||\varphi(t-d)||^2} \\
&\leq K_y n(t)
\end{aligned} \tag{29}
$$

Likewise, from Taylor's theorem there exists a sequence $\overline{u}$ such that

$$
\begin{aligned}
f(y(t), ..., y(t-m), u(t), u(t-1), ..., u(t-r)) &= \\
& f(y(t), ..., y(t-m), 0, u(t-1), ..., u(t-r)) \\
& + \frac{\partial f}{\partial u(t)}(y(t), ..., y(t-m), \overline{u}(t), u(t-1), ..., u(t-r)) u(t)
\end{aligned}
$$

This gives

$$
\begin{aligned}
u(t) = & \left(\frac{\partial f}{\partial u(t)}(y(t), ..., y(t-m), \overline{u}(t), u(t-1), ..., u(t-r)) \right)^{-1} \\
& \cdot (f(y(t), ..., y(t-m), u(t), u(t-1), ..., u(t-r)) \\
& \quad - f(y(t), ..., y(t-m), 0, u(t-1), ..., u(t-r)))
\end{aligned}
$$

and

$$
|u(t)| \leq \beta_0^{-1} 2 K_y n(t) \tag{30}
$$

Hence, from (28), (29) and (30)

$$
\begin{aligned}
\nu(t) &\leq \sum_{i=0}^{\infty} \left(\varepsilon_y K_y |\alpha_i| + \varepsilon_u K_u |\beta_i| \right) n(t-i) + \varepsilon_w n(t) \\
&\leq \sum_{i=0}^{\infty} \left(\varepsilon_y K_y |\alpha_i| + \varepsilon_u K_u |\beta_i| \right) \sqrt{\delta}^{-i} n(t) + \varepsilon_w n(t)
\end{aligned}
$$

and the stated result follows.

□

In the above lemma, the unmodeled dynamics are modeled as impulse responses that are bounded by exponentially decaying sequences, which is a quite general class of unmodeled dynamics that ensures that assumption AU is satisfied.

6.2. Uniform Approximation

Some classes of functions that can approximate arbitrary continuous function to an arbitrary uniform accuracy on compact subsets include radial basis-functions [44], operating regime based models [26], piecewise constant or linear models and splines, fuzzy models [32], and neural networks [13,3,10,65]. Unfortunately, these approximation properties are of very limited practical value. One reason is that the approximation results only hold on compact sets, which means that the system inputs and outputs must be known to be bounded a priori, which significantly reduces the relevance of these approximation results in the analysis of stability. One way that at least in principle resolves this problem is to apply a supervisory control system, like for instance a sliding mode controller [46,55] to ensure that the system state remains within a compact subset of the state-space. Another reason for the inadequacy of uniform approximation results to address the issue of stability in closed loop control systems is related to the number of adjustable parameters required to approximate a given function. Without significant prior knowledge, the number of parameters in the generic model structures mentioned above may be prohibitively large in order to guarantee a uniformly accurate model in a sufficiently large state-space region. It is easy to see that the performance bounds in Theorems 2 and 3 scales with the number of parameters, in the sense that an over-parameterized model may lead to poor robustness and performance bounds. It is therefore desirable to employ a low complexity model structure that matches the structure of the system well. This seems difficult to satisfy, unless significant prior knowledge is applied to choose the function f and the parameter set Θ. The fact that a given class of function approximators have good approximation properties is certainly not sufficient, in general.

Before we proceed with a further discussion of these assumptions, let us first briefly review function approximation using the abovementioned representations. With radial basis-functions, the function f is represented as

$$f(\psi, \theta) \quad = \quad \sum_{i=1}^{N} \theta_i \rho(||\psi - \psi_i||)$$

where the information vector is

$$\psi(t) \;=\; (y(t), ..., y(t-m), u(t), ..., u(t-r))^T$$

and $\rho(\cdot)$ is a given scalar function, e.g. an Gaussian, spline, or multi-quadric function [17,19]. Closely related are normalized radial basis-functions [41] of the form

$$f(\psi, \theta) \;=\; \sum_{i=1}^{N} \theta_i w_i(\psi)$$

where the basis-function

$$w_i(\psi) \;=\; \frac{\rho(||\psi - \psi_i||)}{\sum_{j=1}^{N} \rho(||\psi - \psi_j||)}$$

form a convex combination of the parameter values, since $\sum_{i=1}^{N} w_i(\psi) = 1$ for all ψ. It is assumed that

$$\sum_{i=1}^{N} \rho(||\psi - \psi_i||) \;>\; 0$$

for all ψ. Operating regime based models [29] (also closely related to adaptive connectionist spline networks [31], some fuzzy models [57], and mixtures of local experts [24]) can also be viewed as generalizations of normalized radial basis-function expansions, since they typically take the form

$$f(\psi, \theta) \;=\; \sum_{i=1}^{N} (\theta_{0,i} + \theta_{1,i}^T \psi) w_i(\psi)$$

which leads to a convex combination of linear ARX models. In other words, the constant coefficients in the normalized radial basis-function expansion have been replaced by local linear models.

Finally, consider sigmoidal basis-function models (usually called neural networks) on the form

$$f(\psi, \theta) \;=\; \theta_0 + \sum_{i=1}^{N} \theta_{1,i} \sigma\left(\theta_{2,i} + \theta_{3,i}^T \psi\right)$$

where the function $\sigma(\cdot)$ is sigmoidal (soft saturation function). Contrary to the approximation methods above, the sigmoidal basis-function is non-linearly parameterized.

Models based on all the abovementioned generic function approximation methods satisfy the exponential growth condition in assumption AB. On the other hand, it is in general difficult to design a convex set Θ such that (3) holds, using only knowledge of β_0. Of course, significant prior knowledge is also used to restrict Θ such that one can guarantee that $\overline{\theta}$ is sufficiently small in the case of non-linear parameterization.

7. Extensions and Modifications

In this section we discuss some modifications and extensions of the adaptive control scheme analyzed in this paper. The aim is to develop procedures that are more suitable for practical applications.

7.1. *Extensions to the MIMO case*

Although the theory has been developed for the SISO case, the extension to the MIMO case is conceptually straightforward, provided the MIMO predictor is written in the form

$$
\begin{aligned}
y_1(t+d_1) &= f_1(y(t),...,y(t-m),u(t),...,u(t-r),\theta^\star(t)) + \nu_1(t+d_1) \\
&\vdots \\
y_{m^\star}(t+d_{m^\star}) &= f_{m^\star}(y(t),...,y(t-m),u(t),...,u(t-r),\theta^\star(t)) + \nu_{m^\star}(t+d_{m^\star})
\end{aligned}
$$

where the dimension of the input and output vectors $u(t)$ and $y(t)$ are $r^\star$ and $m^\star$, respectively. The vector relative degree is $d = (d_1, ..., d_{m^\star})$.

The certainty equivalence feedback linearizing control equation is now

$$
\begin{aligned}
v_1(t) &= f_1(y(t),...,y(t-m),u(t),...,u(t-r),\hat{\theta}(t)) \\
&\vdots \\
v_{m^\star}(t) &= f_{m^\star}(y(t),...,y(t-m),u(t),...,u(t-r),\hat{\theta}(t))
\end{aligned}
\tag{31}
$$

With straightforward modifications of the assumptions, the analysis in this paper can easily be extended to also cover the MIMO case, using similar techniques as in [25]. Of course, in the MIMO case, the results will typically be even more conservative and qualitative. This will in particular concern systems that are difficult to decouple robustly, like for instance ill-conditioned systems.

7.2. *Reformulating the Feedback Linearization Control Equation*

Although Assumption AB ensures existence of a solution of the certainty equivalence feedback linearizing control law (26), the actual computation of the control input $u(t)$ is a non-trivial problem, unless this equation is affine in the input $u(t)$. Moreover, Assumption AB is quite restrictive since it requires the sign of the high frequency gain to be constant and bounded from below. Hence, it excludes input saturation and systems where the gain changes sign with the operating point, for instance. Finally, the design of a parameter set

Θ that simultaneously ensures that Assumption AB is satisfied and also contains a sequence of slowly time-varying nominal parameters $\theta^\star(t)$ that provide a sufficiently accurate nominal model is a very difficult problem.

The abovementioned problems suggests an alternative approach, namely approximate feedback linearization. The idea is to develop an algorithm for computing a control input $u(t)$ with the property that it approximates the exact adaptive feedback linearizing control law, simplifies the design of the parameter set Θ, and avoids the restrictive assumptions made above.

The design of approximate feedback linearization controllers has been extensively addressed in the framework of geometric nonlinear control theory [22] for the case of continuous-time systems. Although geometric nonlinear control theory provides powerful methods for controller design, an important class of nonlinear systems do not satisfy the restrictive conditions for feedback linearization. Several approaches have been proposed for designing stable control systems that are based on approximate feedback linearization techniques. These approaches include the pseudo-linearization [52,62], extended linearization [4], linearization families [61], and high-order approximate linearization [63].

On the basis of the MIMO adaptive feedback linearizing control law (31), we suggest the following optimization formulation of the approximate adaptive feedback linearizing control problem:

$$\text{Minimize} \qquad J(u(t)) = \left|\left|v(t) - f(y(t), ..., y(t-m), u(t), ..., u(t-r), \hat{\theta}(t))\right|\right|^2 + \lambda_1 ||u(t)||^2 + \lambda_2 ||u(t) - u(t-1)||^2$$

subject to the constraints

$$\begin{aligned} \underline{u} &\leq u(t) \leq \overline{u} \\ \underline{\Delta u} &\leq u(t) - u(t-1) \leq \overline{\Delta u} \end{aligned}$$

where λ_1 and λ_2 are non-negative constants. We see that when $\lambda_1 = \lambda_2 = 0$ and the constraints are not violated, the solution of the problem of minimizing $J(u(t))$ coincides with the solution of (31).

The above optimization formulation has the following advantages over the exact formulation (31):

- Saturation and constraints on the rate of change of the control input are incorporated in the formulation. More general constraints on inputs and outputs can, of course, be included as well. Hence, the formulation of the control objective is similar to the model predictive control formulation, which has proven very engineering friendly [51].

- By analogy to the linear control case [12], we expect that the use of penalty terms prevents a direct cancellation of the zero dynamics. Hence, it is not always necessary that the inverse nominal model be stable to ensure internal stability of the control system.
- It is no longer necessary to restrict Θ to avoid singularity of the implicit certainty equivalence feedback linearizing control law. The reason is that the quadratic penalty terms in $J(u(t))$ regularizes the inversion of the implicit control law, in the sense that there always exists a $u(t)$ that minimizes $J(u(t))$. In other words, close to singular operating points where the sensitivity of the output $y(t+d)$ with respect to the input $u(t)$ is very small, a more or less arbitrary small input value $u(t)$ is chosen. Because of the low sensitivity at such operating points, this introduces only a small difference on the output compared to the exact feedback linearizing approach.
- One can handle over- or under-determined control problems, i.e. $m^\star \neq r^\star$.

Hence, many of the difficult design problems are resolved with this approach, and new features are added. Unfortunately, the theoretical properties developed in previous sections remains to be extended. Moreover, the problem of actually computing the control input $u(t)$ is not simplified. In general, a non-linear programming algorithm must be applied, and convergence to a global minimum cannot be guaranteed.

7.3. Robust Parameter Estimation

For some problems, a least squares type parameter estimator algorithm may be better suited than the one suggested above, for instance in order to improve the convergence rate. In order to reduce phenomena such as bursting and parameter drift, some further modifications of the parameter estimator may be needed. This can be achieved in different ways using for instance directional forgetting [18,54,45].

8. Concluding Remarks

We have derived theoretical results for an adaptive feedback linearizing control structure for a general class of non-linear systems. The results address some stability, robustness, and performance issues.

The analysis is based on standard input/output stability theory, and in particular the use of weighted l_2-norms. The analysis is completely analogous to the case with a linear model [14]. It is interesting to observe that the complexity of the analysis is only slightly higher compared to the linear case.

The assumptions are in general weak, but difficult to verify. Hence, the results are best interpreted in a qualitative way. Consequently, their immediate practical implications are not very significant, since one in practise is more concerned with quantitative analysis of performance and robustness, rather than a qualitative one. However, the qualitative analysis is still useful, since it illustrates the fundamental limitations and general design trade-offs in a transparent way.

Practical modifications, which are hard to analyze, must often be introduced. These are in particular related to solvability of the certainty equivalence feedback linearizing control law, as well as the need to explicitly take into account fundamental limitations like saturation and non-minimum phase phenomena. Moreover, the on-line parameter estimation algorithm should be modified to improve its robustness.

In practise, significant empirical or mechanistic a priori knowledge must be available for modeling and adaptive controller design. This does not exclude the use of neural networks or fuzzy models in adaptive controllers, but it means that a priori knowledge should be incorporated directly in the neural or fuzzy model like in [57,23,26]. Alternatively, a non-linear black-box model can be combined with for instance a known model, where the non-linear black-box model accounts for the dynamics that are not captured by the known model [33,28,58,50,59].

9. References

1. K. J. Åstrøm, P. Hagander, and J. Sternby. Zeros of sampled systems. *Automatica*, 20:31–38, 1984.

2. A. Baños, F. N. Bailey, and M. A. Armada. Factorization theory and feedback linearization of a class of non-linear systems. In *Proc. IEEE Conf. Decision and Control, Tucson, Az.*, pages 2023–2026, 1992.

3. A. R. Barron. Universal approximation bounds for superpositions of a sigmoidal function. *IEEE Trans. Information Theory*, 39:930–945, 1993.

4. W. T. Baumann and W. J. Rugh. Feedback control of nonlinear systems by extended linearization. *IEEE Trans. Automatic Control*, 31:40–46, 1986.

5. C. I. Byrnes and A. Isidori. A frequency domain philosophy for nonlinear systems. In *Proceedings of the 1984 IEEE Conf. Decision and Control, Las Vegas, Nevada*, pages 1569–1573, 1984.

6. C. I. Byrnes and A. Isidori. Local stabilization of minimum phase nonlinear systems. *Systems and Control Letters*, 11:9–17, 1988.

7. F.-C. Chen and H. K. Khalil. Adaptive control of nonlinear systems using neural networks - a dead-zone approach. In *Proc. American Control Conference, Boston*, pages 667–672, 1991.

8. F.-C. Chen and H. K. Khalil. Adaptive control of nonlinear systems using neural networks. *Int. J. Control*, 55:1299–1317, 1992.

9. F.-C. Chen and C.-C. Liu. Adaptively controlling nonlinear continuous-time systems using multilayer neural networks. *IEEE Trans. Automatic Control*, 39:1306–1310, 1994.

10. T. Chen and H. Chen. Approximation of continuous functionals by neural networks with application to dynamical systems. *IEEE Trans. Neural Networks*, 4:910–918, 1993.

11. D. W. Clarke. Self-tuning control of non-minimum phase systems. *Automatica*, 20:501–517, 1984.

12. D. W. Clarke and P. J. Gawthrop. A self-tuning controller. *IEE Proc. D*, 122:927–934, 1975.

13. G. Cybenko. Approximations by superpositions of a sigmoidal function. *Mathematics of Control, Signals and Systems*, 2:303–314, 1989.

14. A. Datta. Robustness of discrete-time adaptive controllers: An input-output approach. *IEEE Trans. Automatic Control*, 38:1852–1857, 1993.

15. C. A. Desoer and M. G. Kabuli. Stability and robustness of non-linear unity-feedback system: Factorization approach. *Int. J. Control*, 47:1133–1148, 1988.

16. C. A. Desoer and M. Vidyasagar. *Feedback Systems: Input-Output Properies.* Academic Press, New York, 1975.

17. N. Dyn. Interpolation and approximation by radial and related functions. In C. K. Chui, L. L. Schumaker, and D. J. Ward, editors, *Topics in multivariate approximation.* Academic Press, New York, NY, 1987.

18. T. R. Fortescue, L. S. Kershenbaum, and B. E. Ydstie. Implementation of self-tuning regulators with variable forgetting factors. *Automatica*, 17:831–835, 1981.

19. F. Girosi, M. Jones, and T. Poggio. Priors, stabilizers and basis functions: From regularization to radial, tensor and additive splines. Technical Report AI Memo 1430, MIT, Cambridge, 1994.

20. J. Hammer. Non-linear systems, stabilization and coprimeness. *Int. J. Control*, 42:1–20, 1985.

21. P. Ioannou and A. Datta. Robust adaptive control: A unified approach. *Proceedings of the IEEE*, 79:1736–1768, 1991. Also in *Foundations of Adaptive Control*, P. V. Kokotović, Ed., Springer-Verlag, Berlin, 1991.

22. A. Isidori. *Nonlinear Control Systems, 2nd Ed.* Springer Verlag, Berlin, 1989.

23. R. A. Jacobs and M. I. Jordan. Learning piecewise control strategies in a modular neural network architecture. *IEEE Trans. Systems, Man, and Cybernetics*, 23:337–345, 1993.

24. R. A. Jacobs, M. I. Jordan, S. J. Nowlan, and G. E. Hinton. Adaptive mixtures of local experts. *Neural Computation*, 3:79–87, 1991.

25. T. A. Johansen. Fuzzy model based control: Stability, robustness, and performance issues. *IEEE Trans. Fuzzy Systems*, 2:221–234, 1994.

26. T. A. Johansen. Operating regime based process modeling and identification. Technical Report 94-109-W. Dr. Ing. thesis, Department of Engineering Cybernetics, Norwegian Institute of Technology, Trondheim, Norway, 1994. http://www.itk.unit.no/ansatte/Johansen,Tor.Arne/dring_taj.ps.gz.

27. T. A. Johansen. Weighted l_2-norms for analysis of an adaptive control loop based on a non-linear model. Accepted for IEEE Conf. Decision and Control, New Orleans, LA, 1995.

28. T. A. Johansen and B. A. Foss. Representing and learning unmodeled dynamics with neural network memories. In *Proceedings of the American Control Conference, Chicago, Il.*, pages 3037–3043, June 1992.

29. T. A. Johansen and B. A. Foss. Constructing NARMAX models using ARMAX models. *Int. J. Control*, 58:1125–1153, 1993.

30. T. A. Johansen and P. A. Ioannou. Robust adaptive control of minimum phase nonlinear systems. Accepted for publication in Int. J. Adaptive Control and Signal Processing, 1995.

31. R. D. Jones and co-workers. Nonlinear adaptive networks: A little theory, a few applications. Technical Report 91-273, Los Alamos National Lab., NM, 1991.

32. B. Kosko. Fuzzy systems as universal approximators. *IEEE Trans. Computers*, 43:1329–1333, 1994.

33. M. A. Kramer, M. L. Thompson, and P. M. Phagat. Embedding theoretical models in neural networks. In *Proceedings American Control Conference, Chicago, Il.*, pages 475–479, 1992.

34. G. Kreisselmeier. Adaptive control of a class of slowly time-varying plants. *Systems and Control Letters*, 8:97–103, 1986.

35. G. Kreisselmeier and B. D. O. Anderson. Robust model reference adaptive control. *IEEE Trans. Automatic Control*, 31:127–133, 1986.

36. I. J. Leontaritis and S. A. Billings. Input-output parametric models for non-linear systems. *Int. J. Control*, 41:303–344, 1985.

37. F. L. Lewis, K. Liu, and A. Yesildirek. Multilayer neural net robot controller with guaranteed tracking performance. In *Proc. American Control Conference, San Francisco*, pages 2785–2791, 1993.

38. R. Marino and P. Tomei. Global adaptive output-feedback control of nonlinear systems, Part II: Nonlinear parametrization. *IEEE Trans. Automatic Control*, 38:33–49, 1993.

39. R. H. Middleton and G. C. Goodwin. Adaptive control of time-varying linear systems. *IEEE Trans. Automatic Control*, 33:150–155, 1988.

40. S. Monaco and D. Normand-Cyrot. Minimum-phase nonlinear discrete-time systems and feedback stabilization. In *Proc. IEEE Conf. Decision and Control, Los Angeles, Ca.*, pages 979–986, 1987.

41. J. Moody and C. J. Darken. Fast learning in networks of locally-tuned processing units. *Neural Computation*, 1:281–294, 1989.

42. K. S. Narendra and A. M. Annaswamy. *Stable Adaptive Systems.* Prentice-Hall, Englewood Cliffs, NJ, 1989.

43. H. Nijmeijer and A. J. van der Schaft. *Nonlinear Dynamical Control Systems.* Springer-Verlag, New York, 1990.

44. J. Park and I. W. Sandberg. Approximation and radial-basis-function networks. *Neural Computation*, 5:305–316, 1993.

45. J. E. Parkum, N. K. Poulsen, and J. Holst. Selective forgetting in adaptive procedures. In *Proc. 11th IFAC World Congress, Tallin, Estonia*, 1990.

46. M. M. Polycarpou and P. A. Ioannou. Identification and control of nonlinear systems using neural network models: Design and stability analysis. Technical Report 91-09-01, Dept. Electrical Enginering – Systems, University of Southern California, Los Angeles, September 1991.

47. M. M. Polycarpou and P. A. Ioannou. Modeling, identification and stable adaptive control of continuous-time nonlinear dynamical sytems using neural networks. In *Proc. American Control Conference, Chicago*, pages 36–40, 1992.

48. M. M. Polycarpou and P. A. Ioannou. Neural networks as on-line approximators of nonlinear system. In *Proc. 31st IEEE Conf. Decision and Control, Tucson, Az.*, pages 7–12, 1992.

49. L. Praly. Robust model reference adaptive controllers, Part I: Stability analysis. In *Proc. IEEE Conf. Decision and Control, Las Vegas, Nevada*, pages 1009–1014, 1984.

50. D. C. Psichogios and L. H. Ungar. A hybrid neural network - first principles approach to process modeling. *AIChE J.*, 38:1499–1511, 1992.

51. J. B. Rawlings, E. S. Meadows, and K. R. Muske. Nonlinear model predictive control: A tutorial and survey. In *Preprints IFAC Symposium ADCHEM, Kyoto, Japan*, pages 203–214, 1994.

52. C. Reboulet and C. Champetier. A new method for linerizing non-linear systems: The pseudolinearization. *Int. J. Control*, 40:631–638, 1984.

53. W. Rudin. *Real and Complex Analysis.* McGraw-Hill, New York, 1986.

54. S. Sælid, O. Egeland, and B. Foss. A solution to the blow-up problem in adaptive controllers. *Modeling, Identification and Control*, 6(1):39–56, 1985.

55. R. M. Sanner and J.-J. Slotine. Gaussian networks for direct adaptive control. *IEEE Trans. Neural Networks*, 3:837–863, 1992.

56. S. S. Sastry and A. Isidori. Adaptive control of linearizable systems. *IEEE Trans. Automatic Control*, 34:1123–1131, 1989.

57. T. Takagi and M. Sugeno. Fuzzy identification of systems and its application to modeling and control. *IEEE Trans. Systems, Man, and Cybernetics*, 15:116–132, 1985.

58. S. Tan. A combined PID and neural control scheme for nonlinear dynamical systems. In *Proceeding of the IEEE Int. Conf. Intelligent Control and Instrumentation, Singapore*, volume 1, pages 377–383, 1992.

59. M. L. Thompson and M. A. Kramer. Modeling chemical processes using prior knowledge and neural networks. *AIChE J.*, 40:1328–1340, 1994.

60. K. S. Tsakalis. Robustness of model reference adaptive controllers: Input-Output properties. *IEEE Trans. Automatic Control*, 37:556–565, 1992.

61. J. Wang and W. J. Rugh. Feedback linearization families for nonlinear systems. *IEEE Trans. Automatic Control*, 32:935–940, 1987.

62. J. Wang and W. J. Rugh. On the pseudo-linearization problem for nonlinear systems. *Systems and Control Letters*, 12:161–167, 1989.

63. Z. Xu and J. Hauser. Higher order approximate feedback linearization about a manifold. *J. Math Syst., Estimation and Control*, 4:451–465, 1994.

64. B. E. Ydstie. Transient performance and robustness of direct adaptive control. *IEEE Trans. Automatic Control*, 37:1091–1105, 1992.

65. R. Żbikowski and A. Dzieliński. Neural approximation: A control perspective. In K. Kunt, G. Irwin, and K. Warwick, editors, *Neural Network Engineering in Dynamic Control Systems*, pages 1–26. Springer-Verlag, London, 1995.