CHARACTERIZATION OF LYAPUNOV FUNCTIONS FOR SMOOTH NONLINEAR SYSTEMS USING LMIS

Tor A. Johansen *

* *Department of Engineering Cybernetics, Norwegian University of Science and Technology, N-7034 Trondheim, Norway. Email: Tor.Arne.Johansen@itk.ntnu.no*

Abstract: A generic parameterization of Lyapunov function candidates is suggested for a quite general class of non-autonomous smooth nonlinear systems. It is shown that conditions for the existence of a Lyapunov function that guarantees uniform exponential stability can be formulated as a set of linear matrix inequalities. Hence, computation of the Lyapunov function involves the solution of a convex optimization problem for which reasonably efficient algorithms exist. The tradeoff between accuracy and computational complexity as well as possible conservativeness of the procedure is given particular attention.

Keywords: Nonlinear systems; stability; linear matrix inequalities.

1. INTRODUCTION

This work describes a procedure for computing a Lyapunov functions (if one exists) for the equilibrium point $x = x_0$ for the class of non-autonomous non-linear systems that can be written in the form

$$\dot{x} = A(x, \theta)(x - x_0) \tag{1}$$

where $x \in R^n$ is the state vector, $\theta \in R^d$ is a possibly time-varying parameter vector, and the $n \times n$ matrix-valued function $A : R^n \times R^d \rightarrow R^{n \times n}$ is smooth.

There are significant amounts of research on quadratic stability (i.e. conditions for the existence of a quadratic Lyapunov function) of closely related classes of systems, e.g. (Kamenetskii and Pyatnitskii 1987, Becker and Packard 1994, Boyd *et al.* 1994, Gahinet *et al.* 1995). Indeed, restriction to the class of quadratic Lyapunov function candidates may lead to significant conservativeness. In particular, linear parameter varying (LPV) systems $\dot{x} = A(\theta)x$ (where θ is time-varying but known in real time) have recently attracted great interest, and in addition to the quadratic stability results mentioned above, there are stability conditions available that utilize quadratic parameter-dependent Lyapunov-functions of the form $V(x) = x^T P(\theta) x$ and information on the rate of variation of the parameter θ to reduce the conservativeness (Gahinet *et al.* 1996, Wu *et al.* 1996, Watanabe *et al.* 1996). However, in many practical cases $A(\cdot)$ depends on the states rather than external parameters. To embed nonlinear systems into the LPV framework, the states that $A(\cdot)$ depend on are typically being viewed as time-varying parameters and the nonlinear dynamics characterized by a uniform bound on $\dot{\theta}$ (Gahinet *et al.* 1996, Wu *et al.* 1996, Watanabe *et al.* 1996). This may obviously lead to conservative results when utilizing these procedures. The purpose of the present work is to utilize the explicit knowledge of the dependence of $A(\cdot)$ on x in our formulation to reduce the conservativeness by introducing a parameterized class of non-quadratic Lyapunov functions. The procedure presented in this paper reduces the problem of checking the existence of a suitable Lyapunov function of the form $V(x) = (x - x_0)^T P(x)(x - x_0)$ to solving a state-dependent matrix inequality (SDMI) that can be approximated by a finite set of linear matrix inequalities (LMIs). Hence, reasonably efficient convex optimization algorithms (Boyd *et al.* 1994, Gahinet *et al.* 1995) can be applied. This characterization of the Lyapunov function is related to the SDMI characterizations of H_∞ optimal controllers studied in (Lu and Doyle 1995). However, the present approach to approximating the SDMI characterization by a finite number of LMIs differs considerably from the computational procedures suggested in the cited work.

Introducing a parameterized Lyapunov-function is certainly not a new idea, e.g. (Brayton and

Tong 1980, Michel *et al.* 1984, Borisov and Diligenskii 1986, Molchanov and Pyatnitskii 1986, Kamenetskii and Pyatnitskii 1987, Molchanov 1987, Barabanov 1988, Barabanov 1989, Ohta *et al.* 1993, Zelentsovsky 1994, Gahinet *et al.* 1996, Wu *et al.* 1996, Johansson and Rantzer 1998, Petterson and Lennartson 1997, Watanabe *et al.* 1996). The contribution of the present procedure is the utilization of a generic parameterization that does not introduce significant conservativeness and allows the problem to be reduced to a convex optimization problem involving LMI constraints. This leads to a tight and efficient computational procedure for systems of sufficiently low order.

2. LMI CONDITIONS FOR UNIFORM EXPONENTIAL STABILITY

Uniform exponential stability of the system (1) is considered. In other words, let $X^0 \subset R^n$ be a compact and convex region of the state space such that $x_0 \in X^0$, and $\Theta \subset R^d$ be a compact region of the parameter space. A Lyapunov function that guarantees that $x(0) \in X \subset X^0$ implies $x(t) \to x_0$ as $t \to \infty$ at an exponential rate for any parameter trajectory that satisfies $\theta(t) \in \Theta$ for all t is sought. Knowledge of θ is not assumed to be available.

Consider a Lyapunov function candidate $V : X^0 \to R$ of the form

$$V(x) = (x - x_0)^T P(x)(x - x_0) \tag{2}$$

where the symmetric matrix valued function $P : X^0 \to R^{n\times n}$ is defined by the following linear parameterization

$$P(x) = \sum_{i=1}^{N} P_i \rho_i(x) \tag{3}$$

where $P_i = P_i^T > 0$ for all $i = 1, 2, ..., N$, and $\rho_i : X^0 \to R$ are smooth basis-functions. The attention is restricted to positive semi-definite basis-functions that form a partition-of-unity:

$$\sum_{i=1}^{N} \rho_i(x) = 1, \text{ for all } x \in X^0 \tag{4}$$

These conditions ensure that V is indeed a Lyapunov function candidate since it satisfies $V(x_0) = 0$, $V(x) > 0$ for $x \in X^0 - \{x_0\}$, and it is smooth.

The time-derivative of V along the trajectory of the system for a given parameter trajectory is

$$\begin{aligned}\frac{d}{dt}V(x) = (x - x_0)^T &\left(A^T(x,\theta)P(x) + P(x)A(x,\theta)\right.\\ &\left.+\frac{d}{dt}P(x)\right)(x - x_0)\end{aligned} \tag{5}$$

A condition for uniform exponential stability is now that there exists a function P such that the matrix-valued function $L : X^0 \times \Theta \to R^{n\times n}$

$$\begin{aligned}L(x,\theta) &= A^T(x,\theta)P(x) + P(x)A(x,\theta) + \frac{d}{dt}P(x)\\ &= \sum_{i=1}^{N}\left(A^T(x,\theta)P_i\rho_i(x) + P_iA(x,\theta)\rho_i(x)\right.\\ &\qquad \left.+P_i\left(\left(\frac{d\rho_i(x)}{dx}\right)^T A(x,\theta)x\right)\right)\end{aligned}$$

satisfies

$$L(x,\theta) < -\gamma P(x) \tag{6}$$

for some constant $\gamma > 0$ and all $x \in X^0$ and $\theta \in \Theta$. Eq. (6) defines an infinite number of matrix inequalities. Below, a finite discretization of the state and parameter spaces is introduced in order to reduce this condition to a finite number of LMIs.

The following theorem gives conditions on P_1, P_2, ..., P_N that ensures V is a Lyapunov function, and forms the basis for the computational procedure that will be described below.

Theorem 1. Let X^0 be a compact and convex set. Suppose $P_i = P_i^T > 0$ for all $i = 1, 2, ..., N$ and there exists a scalar $\gamma > 0$ such that $L(x,\theta) < -\gamma P(x)$ for all $x \in X^0$ and $\theta \in \Theta$. Then for all parameter trajectories $\theta(t) \in \Theta$ and initial conditions $x(0) \in X$, the equilibrium point $x_0 \in X^0$ is uniformly exponentially stable, i.e.

$$||x(t) - x_0||_2 \le \sqrt{\frac{c_2}{c_1}}||x(0) - x_0||_2 e^{-\gamma t/2} \tag{7}$$

where $c_1 = \min_i \underline{\sigma}(P_i)$, $c_2 = \max_i \overline{\sigma}(P_i)$. The region of attraction X is estimated by

$$X = \left\{x \in X^0 \mid V(x) \le \inf_{\xi\in\partial X^0} V(\xi)\right\} \tag{8}$$

Proof: Since $L(x,\theta) < 0$ on X^0 is is clear that $x(0) \in X$ guarantees $x(t) \in X^0$ for all $t \ge 0$. Now

$$\begin{aligned}\alpha_1(||x - x_0||_2) &= c_1||x - x_0||_2^2 \le V(x)\\ &\le c_2||x - x_0||_2^2 = \alpha_2(||x - x_0||_2)\end{aligned}$$

and

$$\frac{d}{dt}V(x) < -\gamma V(x) = -\alpha_3(||x - x_0||_2) \tag{9}$$

It is clear that α_1, α_2 and α_3 are class $\mathcal{K}$ functions and the result follows from Theorem 4.1 and Corollary 4.2 of (Khalil 1992).

□

3. COMPUTATIONAL PROCEDURE

The SDMI $L(x,\theta) < -\gamma P(x)$ on $X^0 \times \Theta$ defines an infinite number of LMIs in the finite number of parameters $P_1, P_2, ..., P_N$. A finite number of LMIs results from discretization of the compact sets X^0 and Θ by defining finite design sets X_d^0 and Θ_d at which the following constraints on the matrices $P_1, P_2, ..., P_N$

$$L(x,\theta) < -\alpha P(x), \text{ for all } (x,\theta) \in X_d^0 \times \Theta_d \quad (10)$$

are imposed for some $\alpha > 0$. This is a finite number of LMI constraints in a finite number of variables, which are computationally feasible.

However, because of the finite discretization, the existence of positive definite symmetric matrices $P_1, ..., P_N$ that satisfies (10) is not sufficient to guarantee that V is a Lyapunov function. It must be checked that there exists a $\gamma > 0$ such that $L(x,\theta) < -\gamma P(x)$ holds for all $x \in X^0$ and $\theta \in \Theta$. Then it can be argued that $x(0) \in X$ guarantees uniform exponential stability of x_0. In practise, this condition must be checked in a sufficiently dense set of checking points $X_c^0 \times \Theta_c \subset X^0 \times \Theta$. If it does not hold, the density of the set of design points X_d^0 and the parameterization of P should be made richer, and the above procedure iterated.

A computational procedure for searching for a Lyapunov function is:

- Input data: The system function A, a compact and convex set X^0 that contains an interior point x_0, and a compact set Θ.
- Step 1: Select a parameterization of P.
- Step 2: Select finite design sets $X_d^0 \subset X^0$ and $\Theta_d \subset \Theta$.
- Step 3: Solve the convex optimization problem of maximizing α subject to the constraints
$$P_i = P_i^T > 0 \;\text{ and }\; L(x,\theta) < -\alpha P(x)$$
$$\text{for all } x \in X_d^0 \text{ and } \theta \in \Theta_d$$
- Step 4: Generate "sufficiently dense" but finite checking sets $X_c \subset X^0$ and $\Theta_c \subset \Theta$.
- Step 5: If $L(x,\theta) < -\gamma P(x)$ does not hold on $X_c \times \Theta_c$ for some $\gamma > 0$, go to either Step 1 or Step 2.
- Output data: If the procedure converges, a Lyapunov function has been found, and the the region of attraction X can be estimated according to (8).

The above procedure is not an algorithm, in the sense that several of the steps requires further specification. In order to keep the procedure computationally efficient, these steps should rely on engineering insight. "Very dense" uniform grids of design and checking points should be avoided, in particular in high-dimensional state and parameter spaces since they will make the problem computationally intractable. Important information about the required density of design and checking points in the state and parameter spaces as well as the parameterization of P can be determined by analyzing the nonlinearity of A over different regions in the state and parameter spaces. The idea is that if A is a highly nonlinear function in some regions of the state or parameter spaces, it makes sense to allow P to be highly non-quadratic in these regions, and to increase the density of design and checking points in these regions. Define the checking set granularity function $\varepsilon : X^0 \times \Theta \to R$

$$\varepsilon(x,\theta) = \inf_{(\xi,\zeta)\in X_c^0 \times \Theta_c} ||(x,\theta) - (\xi,\zeta)||_2 \quad (11)$$

The above considerations can be formalized theoretically by assuming A, ρ_i and $d\rho_i/dx$ to be locally Lipschitz functions in the sense that for every $(x_1,\theta_1), (x_2,\theta_2)$ in an open ball with radius $\varepsilon(x,\theta)$ around a point (x,θ) there exist bounded functions $L_A : X^0 \times \Theta \to R$, $L_\rho : X^0 \to R$ and $L_{\rho'} : X^0 \to R$ that satisfy

$$\begin{aligned} ||A(x_1,\theta_1) - A(x_2,\theta_2)||_2 \leq \\ L_A(x,\theta)||(x_1,\theta_1) - (x_2,\theta_2)||_2 \\ |\rho_i(x_1) - \rho_i(x_2)| \leq L_\rho(x)||x_1 - x_2||_2 \\ \left|\left|\frac{d\rho_i}{dx}(x_1) - \frac{d\rho_i}{dx}(x_2)\right|\right|_2 \leq L_{\rho'}(x)||x_1 - x_2||_2 \end{aligned}$$

Furthermore, define

$$K_1(x,\theta) = \sup_{(\xi,\zeta)\in B_{\varepsilon(x,\theta)}((x,\theta))} ||A(\xi,\zeta)||_2$$
$$K_2(x,\theta) = \sup_{(\xi,\zeta)\in B_{\varepsilon(x,\theta)}((x,\theta))} ||A(\xi,\zeta)\xi||_2$$

Theorem 2. Suppose X^0 and Θ are compact sets, and there exists an $\alpha > 0$ such that for all $(\xi,\zeta) \in X_c^0 \times \Theta_c$

$$L(\xi,\zeta) < -\alpha P(\xi) \quad (12)$$

Assume A, ρ_i and $d\rho_i/dx$ are locally Lipschitz and assume there exists a $\alpha > \gamma > 0$ such that for all $x \in X^0$ and $\theta \in \Theta$ the checking grid granularity $\varepsilon(x,\theta)$ is so fine that

$$\varepsilon(x,\theta) \leq \frac{\alpha - \gamma}{n(x,\theta)} \quad (13)$$

where

$$\begin{aligned} n(x,\theta) = \kappa(P(x))(2L_A(x,\theta) + (2K_1(x,\theta) + \alpha)L_\rho(x) \\ + K_2(x,\theta)L_{\rho'}(x)) \; > \; 0 \end{aligned}$$

and $\kappa(P) = \overline{\sigma}(P)/\underline{\sigma}(P)$ is the condition number of P. Then for all $x \in X^0$ and $\theta \in \Theta$

$$L(x,\theta) < -\gamma P(x) \quad (14)$$

Proof: Let $(x,\theta) \in X^0 \times \Theta$ be arbitrary. It follows from (12) and (11) that there exists a $(\xi,\zeta) \in (X_c^0 \times \Theta_c) \cap B_{\varepsilon(x,\theta)}((x,\theta))$ such that

$$L(x,\theta) < -\alpha P(x) + Q(x,\xi,\theta,\zeta) \qquad (15)$$

where

$$\begin{aligned} Q(x,\xi,\theta,\zeta) = {} & (A(x,\theta) - A(\xi,\zeta))^T P(\xi) \\ & + P(x)(A(x,\theta) - A(\xi,\zeta)) \\ & + A^T(x,\theta)(P(x) - P(\xi)) \\ & + (P(x) - P(\xi))A(\xi,\zeta) \\ & + \frac{d}{dt}(P(x) - P(\xi)) + \alpha(P(x) - P(\xi)) \end{aligned}$$

It is straightforward to see that $\alpha > \gamma > 0$ and

$$||Q(x,\xi,z,\zeta)||_2 \leq \frac{\alpha - \gamma}{\kappa(P(x))} ||P(x)||_2 \qquad (16)$$

implies

$$-\alpha P(x) + Q(x,\xi,\theta,\zeta) \leq -\gamma P(x) \qquad (17)$$

and the theorem follows from (15) by verifying that the stated local Lipschitz conditions and constraint on the granularity $\varepsilon(x,\theta)$ is sufficient for (16) to hold.

□

This theorem relates the required local checking set granularity to local nonlinearity measures (local Lipschitz constants) of the system function and the Lyapunov function. Introducing more restrictive semi-global Lipschitz conditions a simplified version of this results presents a uniform bound $\overline{\varepsilon}$ on the required granularity of a regular checking set grid:

Corollary 1

Suppose X^0 and Θ are compact sets, and there exists an $\alpha > 0$ such that for all $(\xi,\zeta) \in X_c^0 \times \Theta_c$

$$L(\xi,\zeta) < -\alpha P(\xi) \qquad (18)$$

Furthermore, suppose there exists constants $\overline{L}_A = \sup_{x,\theta} L_A(x,\theta)$, $\overline{L}_\rho = \sup_x L_\rho(x)$, $\overline{L}_{\rho'} = \sup_x L_{\rho'}(x)$, $\overline{K}_1 = \sup_{x,\theta} K_1(x,\theta)$, $\overline{K}_2 = \sup_{x,\theta} K_2(x,\theta)$, $\overline{\kappa} = \sup_x \kappa(P(x))$ and $\overline{\varepsilon} = \sup_{x,\theta} \varepsilon(x,\theta)$ such that

$$\overline{\varepsilon} \leq \frac{\alpha - \gamma}{\overline{\kappa}\left(2\overline{L}_A + (2\overline{K}_1 + \alpha)\overline{L}_\rho + \overline{K}_2 \overline{L}_{\rho'}\right)} \qquad (19)$$

Then for all $x \in X^0$ and $\theta \in \Theta$

$$L(x,\theta) < -\gamma P(x) \qquad (20)$$

□

The parameterization of P is of great practical and theoretical importance in order to find a tight Lyapunov function with a computationally efficient procedure. In order to argue that this parameterization does not introduce significant conservativeness, it is required that the basis-functions can be selected from a set of functions that guarantees that any smooth Lyapunov function candidate and its gradient can be approximated to arbitrary accuracy on X^0. By selecting the basis-functions from a basis that is complete in a Sobolev norm this can be satisfied:

Theorem 3. Suppose X^0 is a compact set and $x_0 \in X^0$. Let $W : X^0 \to R$ be an arbitrary smooth Lyapunov function candidate, i.e. $W(x_0) = 0$, $W(x) > 0$ for any $x \in X^0 - \{x_0\}$. Suppose the set of functions $\mathcal{F}(X^0)$ defines a complete basis for the set of smooth functions in the Sobolev p-norm

$$||f||_{1,p} = ||f||_p + \left|\left|\frac{df}{dx}\right|\right|_p$$

where $p \geq 1$ is arbitrary. Then for any $\delta > 0$ there exist basis-functions $\rho_1, \rho_2, ..., \rho_N \in \mathcal{F}(X^0)$ and matrices $P_1, P_2, ..., P_N$ such that

$$V(x) = \sum_{i=1}^{N} (x - x_0)^T P_i (x - x_0) \rho_i(x) \qquad (21)$$

satisfies

$$||W - V||_p \leq \delta, \quad \left|\left|\frac{dW}{dx} - \frac{dV}{dx}\right|\right|_p \leq \delta$$

□

4. NUMERICAL EXAMPLE

Consider the autonomous non-linear system with equilibrium $x_0 = 0$ defined by the matrix-valued function

$$A(x) = \begin{pmatrix} -3 & 1 \\ \dfrac{2x_1}{0.3 + (x_2 + 0.4)(x_2 - 0.6)} & -2 \end{pmatrix}$$

We apply normalized local basis-functions of the form [1]

$$\begin{aligned} r_i(x) &= \exp\left(-0.5(x - x_i)^T \Sigma_i^{-1}(x - x_i)\right) \\ \rho_i(x) &= \frac{r_i(x)}{\sum_{j=1}^{N} r_j(x)} \end{aligned}$$

and define the subset of the state-space $X^0 = [-1,1] \times [-1,1] \subset R^2$. Consider the following two parameterizations of P:

[1] Notice that this class of basis-functions is closely related to a class for which Sobolev norm approximation properties are known (Rovatti 1996).

$\mathcal{P}_1$: Only one basis-function, which leads to a quadratic Lyapunov function on X^0.

$\mathcal{P}_2$: 4 basis-functions uniformly covering the set X^0.

Next, we select a total of 441 design points regularly distributed over X^0. The LMI problem with constraints (11) does not have a feasible solution with parameterization $\mathcal{P}_1$, while it leads to a Lyapunov function with parameterization $\mathcal{P}_2$, cf. Figure 1, under the assumptions that the checking set is "sufficiently dense" for this conclusion to be made. In Figure 2 $\log_{10}(\varepsilon(x))$ is illustrated from which the uniform bound $\bar{\varepsilon} = 0.4466 \cdot 10^{-3}$ can be found for the choice $\gamma = \alpha/2$. The contributions to the expression (13) for $\varepsilon(x)$ are illustrated in Figure 3. Application of a regular checking grid thus requires a discretization interval $\Delta x_1 = \Delta x_2 = 0.632 \cdot 10^{-3}$ which lead to the order of 10^7 checking point. It is verified for this example that the computed function is indeed a Lyapunov function using this grid size. However, from Figure 2 it is also clear that the checking point density need not be so high in all regions of the state space.

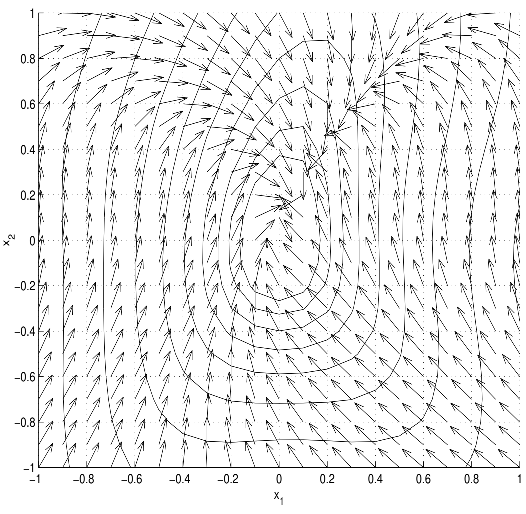

Fig. 1. The vectors (normalized to the same length) illustrates the flow of the dynamic system, while the contour lines illustrate the Lyapunov-function.

From Figure 1 it is clear that the region of attraction is $X = X^0$ (since the flow vector never points out of X^0), although this is not possible to derive from knowledge of the Lyapunov function only. Hence, some conservativeness with respect to the estimates region of attraction must be expected.

5. DISCUSSION AND CONCLUSIONS

A numerical procedure for computation of Lyapunov functions using an LMI solver has been proposed. It is argued that the procedure can be

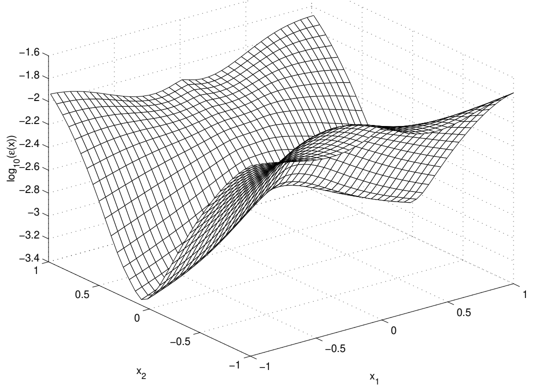

Fig. 2. Checking grid granularity $\varepsilon(x)$.

applied to a quite general class of smooth nonlinear systems.

The applicability of the procedure is somewhat limited by the fact that the number of LMIs will grow exponentially with both the dimension of the state-space and the required accuracy if one applies regular grids of design and checking points. In order to handle high-dimensional systems, the procedure relies on engineering insight in order to avoid regular grids. It is shown that the design and checking points should be concentrated in regions of the state-space where the nonlinearities are most pronounced. A lower bound on the required granularity of the checking point set is characterized in terms of regularity properties of the system function and basis functions in the Lyapunov function parameterization.

For the purpose of checking stability, conservativeness of the approach is due to finite parameterization of the set of Lyapunov function candidates, and finite discretization of the state-space. Also, any prior knowledge on the rate of change of the parameters θ is not required, but combining the present results with the results in (Gahinet *et al.* 1996, Wu *et al.* 1996) should be feasible in order to incorporate this. It would reduce the conservativeness of the method when θ are time-varying parameters known in real time.

The estimated region of attraction may be conservative. The reason for this is that we seek the Lyapunov-function that gives the least conservative estimate of the exponential convergence-rate $\gamma/2$, rather than the Lyapunov-function with the largest region of attraction.

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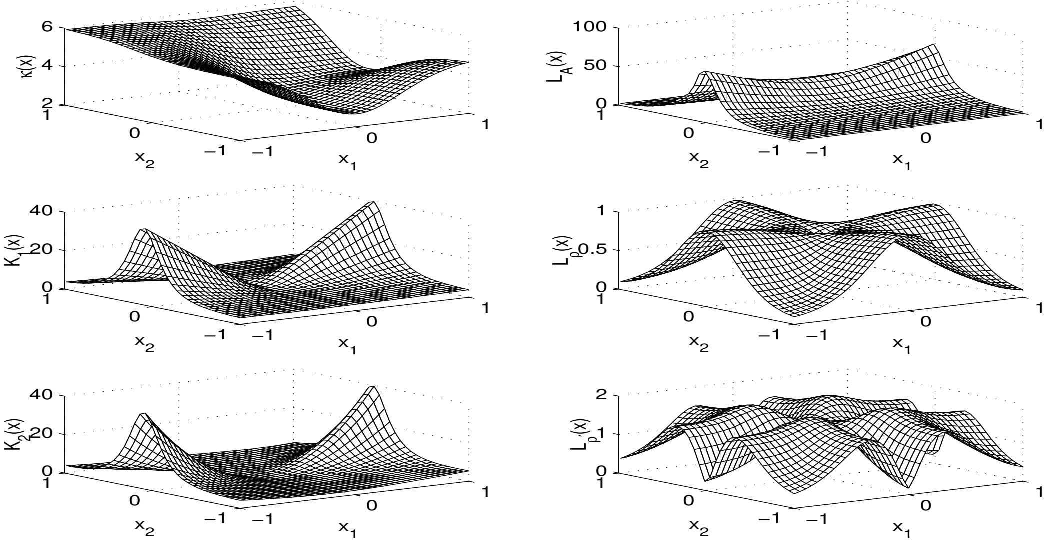

Fig. 3. Contributions to the checking grid granularity.

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