Cascade lemma for set-stable systems

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***Abstract*— A previous result about uniform global asymptotic stability (UGAS) of the equilibrium of a cascaded time-varying systems, is here also shown to hold for closed (not necessarily compact) sets composed by set-stable subsystems of a cascade. In view of this result an optimal control allocation approach is discussed.**

I. Introduction

The work presented in this note is based on a result from [1] where uniform global asymptotic stability (UGAS) of cascaded system

$$\dot{x}_1 = f_1(t, x_1) + g(t, x) \quad (1a)$$
$$\dot{x}_2 = f_2(t, x_2) \quad (1b)$$

where $x := (x_1^{\mathrm{T}}, x_2^{\mathrm{T}})^{\mathrm{T}}$, is concluded for the case $g(t, x) = g_1(t, x)x_2$, if the both origins of $\dot{x}_1 = f_1(t, x_1)$ and $\dot{x}_2 = f_2(t, x_2)$ are UGAS and the solutions of the cascaded system are uniformly globally bounded (UGB). This result motivates a modular approach where the subsystems can be analyzed separately and conclusions about the properties of the cascaded system may be drawn based on the interconnection term $g(t, x)$ instead of analyzing the cascade as one system, which in many cases induces complexity. Background on stability analysis of cascaded systems can be found in [2], [3], [4], [5], [6], [7], [8] and references therein.

Roughly speaking, we show that if a set is uniform global asymptotic stable (UGAS) with respect to a system, then the composed set generated by a cascade of two such systems is itself UGAS under the assumption that the solutions of the cascaded system are UGB with respect to the composed set. We also include a more practical result where UGB is concluded under a certain growth condition.

Our motivation for studying stability properties for sets rests mainly on studying applications that involve solving an optimal control allocation problem. We consider a dynamic control allocation approach that is of special interest for systems that exhibit fast dynamics. This control allocation approach, where stability of optimal sets is addressed, was first considered in [9] and later in [10] by including uncertain parameters, to be adapted, in the effector model. By establishing the result presented here the assumptions in [9] and [10] can be relaxed and a wider class of nonlinear systems may be considered. An idea of how this can be done is shown in section IV.

When the functions f_1, f_2 and g from (1) are locally Lipschitz, this class of nonlinear time-varying systems can

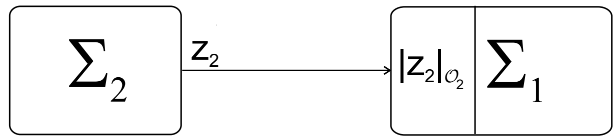

Fig. 1. Cascaded system, where Σ_1 is the *perturbed* system that is assumed UGAS with respect to a set, $\mathcal{O}_1$, as long as $|z_2|_{\mathcal{O}_2} = 0$. Σ_2 is the *perturbing* system.

be seen as a particular case of the following autonomous systems, as far as uniform set-stability is concerned:

$$\dot{z}_1 = F_1(z_1) + G(z) \quad (2a)$$
$$\dot{z}_2 = F_2(z_2) \quad (2b)$$

where $z_1 \in \mathbb{R}^{q_1}$, $z_2 \in \mathbb{R}^{q_2}$, $z := (z_1^{\mathrm{T}}, z_2^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^q$. To see this, just consider the case when $z_1 = (t, x_1^{\mathrm{T}})^{\mathrm{T}}$ and $z_2 = (t, x_2^{\mathrm{T}})^{\mathrm{T}}$ are synchronized by, $\dot{t} = 1$ with same initial conditions for both subsystems. Then, letting $F_1(z_1) = (1, f_1(t, x_1)^{\mathrm{T}})^{\mathrm{T}}$, $G(z) = (0, g(t, x)^{\mathrm{T}})^{\mathrm{T}}$ and $F_2(z_2) = (1, f_2(t, x_2)^{\mathrm{T}})^{\mathrm{T}}$, the cascade (1) takes the form (2). Furthermore, the study of the stability of some given closed (but not necessarily compact) sets $\mathcal{N}_1$ and $\mathcal{N}_2$ for (1) reduces to the study of the stability of $\mathcal{O}_1 := \mathbb{R}_{\geq 0} \times \mathcal{N}_1$ and $\mathcal{O}_2 := \mathbb{R}_{\geq 0} \times \mathcal{N}_2$ for the system (2). This structure of the sets $\mathcal{O}_1$ and $\mathcal{O}_2$ should however be seen as a particular case, and the set-stability properties that we expose in the sequel englobe much more general configurations. The diagram in Figure 1, represents a general cascaded system.

II. Preliminary definitions and results

A. Notation

The function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}$ if it is continuous, strictly increasing and $\alpha(0) = 0$. α is of class $\mathcal{K}_\infty$ if in addition $\alpha(s) \to \infty$ as $s \to \infty$. $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a class $\mathcal{KL}$ function if, for each fixed t, the mapping $\beta(\cdot, t)$ is of class $\mathcal{K}$ and for each fixed s the mapping $\beta(s, \cdot)$ is continuous, decreasing and tends to zero as its argument tends to $+\infty$. $|\cdot|$ denotes the Euclidian norm and $|\cdot|_{\mathcal{A}'} : \mathbb{R}^q \to \mathbb{R}_{\geq 0}$ denotes the distance from a point $z \in \mathbb{R}^q$ to a set $\mathcal{A}' \subset \mathbb{R}^q$, $|z|_{\mathcal{A}'} := \inf \{|z - y| : y \in \mathcal{A}'\}$. The solution of an autonomous dynamic system is denoted by $z(t, z_0)$ where $z_0 = z(t_0, z_0)$ is the initial state. We say that a function $V : \mathbb{R}^q \to \mathbb{R}_{\geq 0}$ is *smooth* if it is infinitely differentiable.

B. Definitions

The definitions that follows are either motivated by or can be found in [11] and [12]. They pertain to systems of the form

$$\dot{z} = F(z)\,, \tag{3}$$

where $F : \mathbb{D} \to \mathbb{R}^q$ is locally Lipschitz with $\mathbb{D} \subset \mathbb{R}^q$.

Definition 1: The system (3) is said to be *forward complete* if, for each $z_0 \in \mathbb{D}$, the solution $z(\cdot, z_0) \in \mathbb{D}$ is defined on $\mathbb{R}_{\geq 0}$.

Definition 2: The system (3) is said to be *finite escape time detectable* through $|\cdot|_{\mathcal{A}}$, if any solution, $z(t, z_0) \in \mathbb{D}$, which is right maximally defined on a bounded interval $[0, T)$, satisfies $\lim_{t \nearrow T} |z(t, z_0)|_{\mathcal{A}} = \infty$.

Definition 3: For the system (3), the closed set $\mathcal{A} \subset \mathbb{D}$ is Uniformly Stable (US) if the system (3) is forward complete, and there exists a function $\nu \in \mathcal{K}$ and a constant $c > 0$ such that, $\forall\ |z_0|_{\mathcal{A}} < c$,

$$|z(t, z_0)|_{\mathcal{A}} \leq \nu(|z_0|_{\mathcal{A}}), \quad \forall t \geq 0\,. \tag{4}$$

Definition 4: For the system (3), when $\mathbb{D} = \mathbb{R}^q$, then the closed set $\mathcal{A} \subset \mathbb{R}^q$ is Uniformly Globally Stable (UGS) if the system (3) is forward complete and (4) is satisfied with $\nu \in \mathcal{K}_\infty$ and for any $z_0 \in \mathbb{R}^q$.

Definition 5: For the system (3), the closed set $\mathcal{A} \subset \mathbb{D}$ is Uniformly Attractive (UA) if the system (3) is forward complete, there exists constant $c > 0$, such that for all $|z_0|_{\mathcal{A}} < c$ and any $\mu > 0$ there exists $T = T(\mu) > 0$, such that

$$|z_0|_{\mathcal{A}} \leq c,\ t \geq T \quad \Rightarrow \quad |z(t, z_0)|_{\mathcal{A}} \leq \mu \tag{5}$$

Definition 6: For the system (3), when $\mathbb{D} = \mathbb{R}^q$, then the closed set $\mathcal{A} \subset \mathbb{R}^q$ is Uniformly Globally Attractive (UGA) if the system (3) is forward complete, and for each pair of strictly positive numbers (c, μ) there exists $T = T(c, \mu) > 0$ such that for all $z_0 \in \mathbb{R}^q$, (5) holds.

Definition 7: For the system (3), the closed set $\mathcal{A}$ is Uniformly Asymptotically Stable (UAS) if it is US and UA.

Definition 8: For the system (3), when $\mathbb{D} = \mathbb{R}^q$, the closed set $\mathcal{A}$ is Uniformly Globally Asymptotically Stable (UGAS) if it is UGS and UGA.

Provided that (3) is forward complete, this definition is well known to be equivalent to the following $\mathcal{KL}$ characterization (see *e.g.* [12], [13]): There exists a class $\mathcal{KL}$ function β such that, for all $z_0 \in \mathbb{R}^q$, $|z(t, z_0)|_{\mathcal{A}} \leq \beta(|z_0|_{\mathcal{A}}, t - 0)$ for all $t \geq 0$.

From [14], we adapt the definition of uniform boundedness of solutions to the case when $\mathcal{A}$ is not reduced to $\{0\}$.

Definition 9: The solutions of system (3) are said to be Uniformly Bounded (UB) with respect to a closed set $\mathcal{A} \subset \mathbb{D}$ if, there exist a positive constant c, such that for every positive constant $r < c$ there is a positive constant $\bar{c} = \bar{c}(r)$, such that

$$|z_0|_{\mathcal{A}} \leq r \quad \Rightarrow \quad |z(t, z_0)|_{\mathcal{A}} \leq \bar{c}\,, \quad \forall t \geq 0\,. \tag{6}$$

Definition 10: The solutions of system (3), where $\mathbb{D} = \mathbb{R}^q$, are said to be Uniformly Globally Bounded (UGB) with respect to a closed set $\mathcal{A} \subset \mathbb{R}^q$ if, for every $r \in \mathbb{R}_{\geq 0}$, there is a positive constant $\bar{c} = \bar{c}(r)$ such that (6) is satisfied.

It can easily be shown that this is equivalent to the existence of a nonnegative constant μ and of a class $\mathcal{K}_\infty$ function η such that, for all $z_0 \in \mathbb{R}^q$,

$$|z(t, z_0)|_{\mathcal{A}} \leq \eta(|z_0|_{\mathcal{A}}) + \mu\,, \quad \forall t \geq 0\,.$$

In what follows, $\mathcal{O}_1 \subset \mathbb{R}^{q_1}$ and $\mathcal{O}_2 \subset \mathbb{R}^{q_2}$ are closed (not necessarily bounded) sets and $\mathcal{A} = \mathcal{O}_1 \times \mathcal{O}_2$.

The set-stability analysis of (2) is done under the following assumptions.

Assumption 1. The functions F_1, F_2 and G are locally Lipschitz.

Assumption 2. The cascade (2) is forward complete.

Assumption 3. There exist a continuous function $G_1 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and a class $\mathcal{K}$ function G_2 such that, for all $z \in \mathbb{R}^q$,

$$|G(z)| \leq G_1(|z|_{\mathcal{A}}) G_2(|z_2|_{\mathcal{O}_2}).$$

Assumption 4. There exists a continuously differentiable Lyapunov function $\bar{V}_1 : \mathbb{R}^{q_1} \to \mathbb{R}_{\geq 0}$, class $\mathcal{K}_\infty$ functions $\bar{\alpha}_1$, $\bar{\alpha}_2$ and $\bar{\alpha}_3$, and a continuous function $\bar{\varsigma} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that, for all $z_1 \in \mathbb{R}^{q_1}$,

$$\bar{\alpha}_1(|z_1|_{\mathcal{O}_1}) \leq \bar{V}_1(z_1) \leq \bar{\alpha}_2(|z_1|_{\mathcal{O}_1}) \tag{7}$$

$$\frac{\partial \bar{V}_1}{\partial z_1}(z_1) F_1(z_1) \leq -\bar{\alpha}_3(|z_1|_{\mathcal{O}_1}) \tag{8}$$

$$\left| \frac{\partial \bar{V}_1}{\partial z_1}(z_1) \right| \leq \bar{\varsigma}\left(|z_1|_{\mathcal{O}_1}\right)\,. \tag{9}$$

It is worth underlining that the existence of a smooth Lyapunov function satisfying (7) and (8) follows directly from [12] or [13] if and only if $\mathcal{O}_1$ is UGAS for $\dot{z}_1 = F_1(z_1)$. However, the bound (9) on the gradient may not be trivial, which justifies this assumption.

Remark 1: For the special case where $z_1 = (t, x_1^{\mathrm{T}})^{\mathrm{T}}$, the bound (9) will be reduced to

$$\left| \frac{\partial \bar{V}_1}{\partial x_1}(z_1) \right| \leq \bar{\varsigma}\left(|z_1|_{\mathcal{O}_1}\right)\,. \tag{10}$$

This is due to $G(z) = (0, g(t, x)^{\mathrm{T}})^{\mathrm{T}}$, and the term of interest in our analysis is: $\frac{\partial \bar{V}_1}{\partial z_1} G(z) \leq \left| \frac{\partial \bar{V}_1}{\partial x_1} \right| |g(t, x)|\,.$

III. Main result

Lemma 1. Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be some closed subsets of $\mathbb{R}^{q_1}$ and $\mathbb{R}^{q_2}$ respectively. Assume that $\mathcal{O}_2$ is UGAS with respect to the system (2b) and that the solutions of system (2) are UGB with respect to $\mathcal{A} := \mathcal{O}_1 \times \mathcal{O}_2$. Then, under Assumptions 1, 2, 3 and 4, the set $\mathcal{A}$ is UGAS for the cascade (2).

Proof: We start by introducing the following result, which borrows from [15, Proposition 13], originally presented in [16]. We have that under Assumption 4, for any nonnegative constant c, there exists a continuously differentiable Lyapunov function $V_1 : \mathbb{R}^{q_1} \to \mathbb{R}_{\geq 0}$, class $\mathcal{K}_\infty$ functions α_1, α_2, and a continuous nondecreasing function $\varsigma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that, for all $z_1 \in \mathbb{R}^{q_1}$,

$$\alpha_1(|z_1|_{\mathcal{O}_1}) \leq V_1(z_1) \leq \alpha_2(|z_1|_{\mathcal{O}_1}) \tag{11}$$

$$\frac{\partial V_1}{\partial z_1}(z_1) F_1(z_1) \leq -c V_1(z_1) \tag{12}$$

$$\left| \frac{\partial V_1}{\partial z_1}(z_1) \right| \leq \varsigma\left(|z_1|_{\mathcal{O}_1}\right) . \tag{13}$$

Let the function $\bar{V}_1$ of Assumption 4 generate a continuously differentiable function V_1 with $c = 1$. In view of Assumption 3, the derivative of V_1 along the solutions of (2a) then yields

$$\dot{V}_1(z_1) \leq -V_1(z_1) + \varsigma(|z_1|_{\mathcal{O}_1}) G_1(|z|_{\mathcal{A}}) G_2(|z_2|_{\mathcal{O}_2}) .$$

From the UGB property, there exist $\mu \geq 0$ and $\eta \in \mathcal{K}_\infty$ such that, for all $z_0 \in \mathbb{R}^q$,

$$|z(t, z_0)|_{\mathcal{A}} \leq \eta(|z_0|_{\mathcal{A}}) + \mu , \quad \forall t \geq 0 . \tag{14}$$

Defining $v(t, z_0) := V_1(z_1(t, z_0))$ and $v_0 := V_1(z_{10})$, we get that [1]

$$\dot{v}(t, z_0) \leq -v(t, z_0) + B(|z_0|_{\mathcal{A}}) G_2(|z_2(t, z_{20})|_{\mathcal{O}_2}) ,$$

where $B(\cdot) := \max\limits_{0 \leq s \leq \eta(\cdot)+\mu} \varsigma(s) G_1(\eta(\cdot)+\mu)$. From the UGAS of (2b) with respect to $\mathcal{O}_2$, there exists $\beta_2 \in \mathcal{KL}$ such that, for all $z_{20} \in \mathbb{R}^{q_2}$,

$$|z_2(t, z_{20})|_{\mathcal{O}_2} \leq \beta_2(|z_{20}|_{\mathcal{O}_2}, t) , \quad \forall t \geq 0 . \tag{15}$$

Accordingly, we obtain that

$$\dot{v}(t, z_0) \leq -v(t, z_0) + \tilde{\beta}(|z_0|_{\mathcal{A}}, t) , \tag{16}$$

where $\tilde{\beta}(r, t) := B(r) G_2(\beta_2(r, t))$. Notice that $\tilde{\beta}$ is a class $\mathcal{KL}$ function. Using that $\tilde{\beta}(|z_0|_{\mathcal{A}}, t - 0) \leq \tilde{\beta}(|z_0|_{\mathcal{A}}, 0)$ and integrating (16) yields, through the comparison lemma,

$$v(t, z_0) \leq v_0 e^{-t} + \tilde{\beta}(|z_0|_{\mathcal{A}}, 0) .$$

It follows that, for all $t \geq 0$,

$$|z_1(t, z_0)|_{\mathcal{O}_1} \leq \alpha_1^{-1}\left(\alpha_2(|z_{10}|_{\mathcal{O}_1}) + \tilde{\beta}(|z_0|_{\mathcal{A}}, 0)\right) ,$$

which, with the UGAS of $\mathcal{O}_2$ for (2b), implies that

$$|z(t, z_0)|_{\mathcal{A}} \leq \nu(|z_0|_{\mathcal{A}}) , \quad \forall t \geq 0 , \tag{17}$$

[1] This is done by noticing that $\max(|z_1|_{\mathcal{O}_1}, |z_2|_{\mathcal{O}_2}) \leq |z|_{\mathcal{A}}$

where $\nu(\cdot) := \sqrt{\alpha_1^{-1}(\alpha_2(\cdot) + \tilde{\beta}(\cdot, 0))^2 + \beta_2(\cdot, 0)^2}$ is a class $\mathcal{K}_\infty$ function. UGS of $\mathcal{A}$ follows.

To prove uniform global attractiveness, consider any positive constants ε_1 and r such that $\varepsilon_1 < r$ and let $T_1(\varepsilon_1, r) \geq 0$ be such that[2] $\tilde{\beta}(r, T_1) = \frac{\varepsilon_1}{2}$, then it follows from the integration of (16) from T_1 to any $t \geq T_1$ that, for any $|z_0|_{\mathcal{A}} \leq r$,

$$\begin{aligned} v(t, z_0) &\leq v(T_1, z_0) e^{-(t-T_1)} + \int_{T_1}^{t} \tilde{\beta}(|z_{20}|_{\mathcal{O}_2}, T_1) e^{-(t-s)} ds \\ &\leq v(T_1, z_0) e^{-(t-T_1)} + \tilde{\beta}(r, T_1)\left(1 - e^{-(t-T_1)}\right) \end{aligned}$$

Consequently, in view of (17),

$$v(t, z_0) \leq \alpha_2 \circ \nu(|z_0|_{\mathcal{A}}) e^{-(t-T_1)} + \frac{\varepsilon_1}{2}$$

Letting $T := T_1 + \ln\left(\frac{2}{\varepsilon_1}\left(\alpha_2 \circ \nu(r) + \tilde{\beta}(r, 0)\right)\right)$ gives $v(t) \leq \varepsilon_1$ for all $t \geq T$. If we define $\varepsilon := \alpha_1^{-1}(\varepsilon_1)$, it follows that $|z_1(t, z_{10})|_{\mathcal{A}} \leq \varepsilon$ for all $t \geq T$. Since ε is arbitrary and $\mathcal{O}_2$ is UGAS for (2b), we conclude that $\mathcal{A}$ is UGA, and the conclusion follows. ■

Corollary 1. Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be some closed subsets of $\mathbb{R}^{q_1}$ and $\mathbb{R}^{q_2}$ respectively. Assume that $\mathcal{O}_2$ is UGS with respect to the system (2b) and that the solutions of system (2) are UGB with respect to $\mathcal{A} := \mathcal{O}_1 \times \mathcal{O}_2$. Then, under Assumptions 1, 2, 3 and 4, the set $\mathcal{A}$ is UGS for the cascade (2).

Proof: From the UGB and UGS property of $\mathcal{A}$ and $\mathcal{O}_2$, (17) is satisfied by noting that $\tilde{\beta}$ in (16) is a class $\mathcal{K}_\infty$ function. ■

In the view of analyzing adaptation strategies and control allocation algorithms, two local results of Lemma 1 are of special interest. In what follows we have $F_2 : \mathbb{D} \to \mathbb{R}^{q_2}$ in (2b) and $G : \mathbb{R}^{q_1} \times \mathbb{D} \to \mathbb{R}^{q_1}$ in (2a).

Corollary 2. Let $\mathcal{O}_1 \subset \mathbb{R}^{q_1}$ and $\mathcal{O}_2 \subset \mathbb{D}$. Assume that $\mathcal{O}_2$ is US with respect to the system (2b). Then, under Assumptions 1, 2, 3 and 4, the set $\mathcal{A} := \mathcal{O}_1 \times \mathcal{O}_2$ is US for the overall cascade (2).

Proof: We first prove that the solutions of the system (2) are UB with respect to $\mathcal{A} := \mathcal{O}_1 \times \mathcal{O}_2$, then we use Lemma 1 to prove the stability result.

From [17] Lemma B.1 there exist continuous functions $B_{z_1} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and $B_{z_2} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, where $B_{z_2}(0) = 0$, such that $\varsigma(|z_1|_{\mathcal{O}_1}) G_1(|z|_{\mathcal{A}}) G_2(|z_2|_{\mathcal{O}_2}) \leq B_{z_1}(|z_1|_{\mathcal{O}_1}) B_{z_2}(|z_2|_{\mathcal{O}_2})$.

From $B_{z_1}(|z_1|_{\mathcal{O}_1})$ being continous, for any $\epsilon_1 > 0$ there exist $\delta_1 > 0$ such that $|z_1|_{\mathcal{O}_1} < \delta_1 \Rightarrow |B_{z_1}(|z_1|_{\mathcal{O}_1}) - B_{z_1}(0)| \leq \epsilon_1$. Fix ϵ_1 and choose ϵ_2 such that there exist a δ_2, by US of $\mathcal{O}_2$, that satisfy $\bar{\alpha}_3^{-1}\left(\left(\epsilon_1 + B_{z_1}(0)\right) B_{z_2}(\delta_2)\right) < \delta_1$ and $\delta_2 < \delta_1$. Then if $\bar{\alpha}_3(|z_{10}|_{\mathcal{O}_1}) < B_{z_1}(|z_{10}|_{\mathcal{O}_1}) B_{z_1}(\delta_2)$:

$$\begin{aligned} \dot{\bar{V}}_1 &\leq -\bar{\alpha}_3(|z_1|_{\mathcal{O}_1}) + B_{z_1}(|z_1|_{\mathcal{O}_1}) B_{z_2}(|z_2|_{\mathcal{O}_2}) \\ &\leq -\bar{\alpha}_3(|z_1|_{\mathcal{O}_1}) + \left(\epsilon_1 + B_{z_1}(0)\right) B_{z_2}(\delta_2) \end{aligned}$$

[2] If $\tilde{\beta}(r, 0) \leq \frac{\varepsilon_1}{2}$, pick T_1 as 0

such that

$$|z_1(t)|_{\mathcal{O}_1} \leq \bar{\alpha}_3^{-1}\left(\left(\epsilon_1 + B_{z_1}(0)\right) B_{z_2}(\delta_2)\right),$$

and $|z(t)|_{\mathcal{A}} \leq c_1$ where

$$c_1 := 2\max(\bar{\alpha}_3^{-1}\left(\left(\epsilon_1 + B_{z_1}(0)\right) B_{z_2}(\delta_2)\right), \delta_2).$$

Else for $\bar{\alpha}_3(|z_{10}|_{\mathcal{O}_1}) \geq B_{z_1}(|z_{10}|_{\mathcal{O}_1}) B_{z_1}(\delta_2)$:

$$|z_1(t)|_{\mathcal{O}_1} \leq \bar{\alpha}_1^{-1}(\bar{\alpha}_2(\delta_2)),$$

and $|z(t)|_{\mathcal{A}} \leq c_2$ where $c_2 := 2\max(\bar{\alpha}_1^{-1}(\bar{\alpha}_2(\delta_2)), \delta_2)$. Thus for all $|z_0|_{\mathcal{A}} \leq \delta_2$, $|z(t)|_{\mathcal{A}} \leq c$, where $c(\delta_2) := \max(c_1, c_2)$, the solutions of system (2) are UB with respect to $\mathcal{A}$.

From the UB and US property of $\mathcal{A}$ and $\mathcal{O}_2$ there exists positive constants c_z and c_{z_2} such that for all $|z_0|_{\mathcal{A}} \leq c_{z_2}$ and $|z_{20}|_{\mathcal{O}_2} \leq c_{z_2}$, (17) is satisfied by notceing that $\tilde{\beta}$ in (16) is, in this case, a class $\mathcal{K}$ function. ■

Corollary 3. Let $\mathcal{O}_1 \subset \mathbb{R}^{q_1}$ and $\mathcal{O}_2 \subset \mathbb{D}$. Assume that $\mathcal{O}_2$ is UAS with respect to the system (2b). Then, under Assumptions 1, 2, 3 and 4, the set $\mathcal{A} := \mathcal{O}_1 \times \mathcal{O}_2$ is UAS for the overall cascade (2).

Proof: By the same arguments as in the proof of Corollary 2, the solutions of system (2) are UB with respect to $\mathcal{A}$.

UB of set $\mathcal{A}$ and UAS of set $\mathcal{O}_2$ imply that there exists some positive constants c_z and c_{z_2}, such that the steps in the proof of Lemma 1 can be followed for some initial condition $|z_0|_{\mathcal{A}} \leq c_z$ and $|z_{20}|_{\mathcal{O}_2} \leq c_{z_2}$. ■

Remark 2: In [18] a result similar to Corollary 3, is proved for the case when the sets $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{A}$ represents the origin of the respective systems.

In most cases the hardest requirement to check when applying Lemma 1 is the uniform global boundedness of the solutions of the cascade with respect to the set $\mathcal{A}$. Inspired by [1], we now propose an alternative to this. More precisely, the following result states that the UGB assumption may be replaced by a simple growth comparison between the $\frac{\partial V_1}{\partial z_1} G(z)$ term and the dissipation rate of V_1 for large values of the state. Its proof is omitted as it follows from minor modifications of that of Theorem 3 in [1]. From [1] we have the "small o" definition:

Definition 11: Let $\varrho(x)$, $\varphi(t,x)$ be continuous functions of their arguments. We denote $\varphi(t,x) = o(\varrho(x))$ if there exists a continuous function $\lambda : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $|\varphi(t,x)| \leq \lambda(|x|)\,|\varrho(x)|$ for all $(t,x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n$ and $\lim_{|x|\to\infty} \lambda(|x|) = 0$.

Assumption 5. For each fixed $z_2 \in \mathbb{R}^{q_2}$, it holds that

$$\left|\frac{\partial V_1}{\partial z_1}(z_1) G(z)\right| = o\left(\alpha_3\left(|z_1|_{\mathcal{O}_1}\right)\right), \quad \text{as } |z_1|_{\mathcal{O}_1} \to \infty\,.$$

Theorem 1. Assume that $\mathcal{O}_2$ is UGAS with respect to the system (2b), then under Assumptions 1, 2, 3, 4 and 5, the set $\mathcal{A} = \mathcal{O}_1 \times \mathcal{O}_2$ is UGAS for the cascade (2).

IV. Motivating example: Dynamic optimizing control allocation

In this section we show how the result can be applied in order to solve an optimizing control allocation problem dynamically. We will not go into the details and technicalities of the problem but rather focus on the idea and problem reformulation. For a complete presentation of the problem see [9] and [10]. It should be noted that for "fast" over-actuated mechanical systems, dynamic control allocation algorithms are of special interest. Stability can be guaranteed and since no numeric optimizing software is needed, implementations on low-cost hardware may be realized with low complexity software. For instance in [19] a yaw stabilization scheme, using a dynamic control allocation algorithm, for an automotive vehicle using brakes is implemented on a realistic simulation environment.

Consider the over-actuated nonlinear system

$$\dot{x} = f(t,x) + g(t,x)\tau \quad \text{(18a)}$$
$$\tau = h(t,x,u) \quad \text{(18b)}$$

where $t \geq 0$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^r$, $\tau \in \mathbb{R}^d$, $d \leq r$, and the functions $f(t,x)$, $g(t,x)$ and $h(t,x,u)$ are locally Lipschitz. Also, $|g(t,x)| \leq \tilde{G}_1(|x|)$ where the function $G_1 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is continuous. Assume that there exists a virtual control $\tau_c := k(t,x)$, such that, $\tau = \tau_c$, uniformly globally asymptotically stabilizes the equilibrium of (18a), then the optimal control allocation problem can be formulated in terms of solving the minimization problem:

$$\min_u J(t,x,u) \quad s.t \quad k(t,x) - h(t,x,u) = 0\,. \quad (19)$$

Instead of looking at an exact "static or quasi-dynamic" optimal solution of (19), we consider a dynamic solution that is related to the first order optimal set of problem (19),

$$\bar{\mathcal{O}}_2(t,x) := \left\{ \left(u^{\mathrm{T}}, \lambda^{\mathrm{T}}\right)^{\mathrm{T}} \in \mathbb{R}^{r+d} \,\middle|\, \left(\frac{\partial \bar{L}}{\partial u}, \frac{\partial \bar{L}}{\partial \lambda}\right)(t,x,u,\lambda) = 0 \right\},$$

by introducing the Lagrangian function

$$\bar{L}(t,x,u,\lambda) = J(t,x_1,u) + (k(t,x) - h(t,x,u))^{\mathrm{T}}\lambda \quad (20)$$

where λ is the Lagrangian multiplier vector.

If we are able to prove that $x_1(t)$ exists for all t, we may represent this problem by a cascade, see Figure 2, where the systems are given by:

$$\Sigma_1 : \begin{cases} \dot{p} = 1 \\ \dot{x} = f(p,x) + g(p,x)k(p,x) + (h(p,x,u) - k(p,x)) \end{cases},$$

$$\Sigma_2 : \begin{cases} \dot{p} = 1 \\ \dot{u} = f_u(p, x(p), u, \lambda) \\ \dot{\lambda} = f_\lambda(p, x(p), u, \lambda) \end{cases}.$$

From the assumption on $k(t,x)$ it is clear that set $\mathcal{O}_1 := \left\{ z_1 := (p, x^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n \,|\, x = 0 \right\}$ is UGAS with respect to Σ_1, as long as $h(t,x,u) = k(t,x)$. Since $\frac{\partial \bar{L}}{\partial \lambda} = k(t,x_1) - h(t,x,u)$ and $\bar{\mathcal{O}}_2(t,x) \subset \bar{\mathcal{O}}_{\partial L \partial u}(t,x) := \left\{ \left(u^{\mathrm{T}}, \lambda^{\mathrm{T}}\right)^{\mathrm{T}} \in \mathbb{R}^{r+d} \,\middle|\, \frac{\partial \bar{L}}{\partial \lambda} = 0 \right\}$, it is also clear that $\left|\left(u^{\mathrm{T}}, \lambda^{\mathrm{T}}\right)^{\mathrm{T}}\right|_{\bar{\mathcal{O}}_2} \geq \left|\left(u^{\mathrm{T}}, \lambda^{\mathrm{T}}\right)^{\mathrm{T}}\right|_{\bar{\mathcal{O}}_{\partial L \partial u}}$. Based

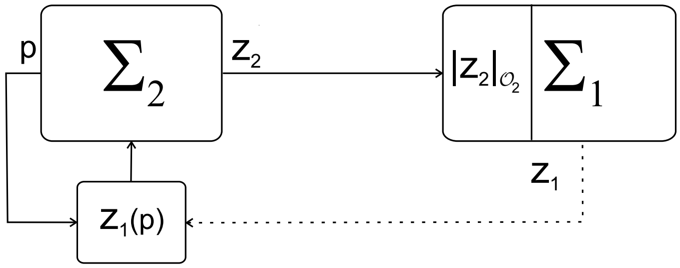

Fig. 2. Σ_2 may be perturbed indirectly by Σ_1 since z_1 may be considered as a time-varying signal, $z_1(t)$, as long as this signal exists for all t.

on this, the task will be to: 1) Construct the update-laws of Σ_2 such that the perturbation of Σ_1, measured with $\left|\left(u^{\mathrm{T}}, \lambda^{\mathrm{T}}\right)^{\mathrm{T}}\right|_{\bar{\mathcal{O}}_2}$, in some sense is UGAS with respect to $\bar{\mathcal{O}}_2$, see [9] and [10]. And 2) prove that the cascade satisfies the assumptions of Lemma 1.

Remark 3: In most mechanical systems there are constraints on the actuators/effectors, which means that $u \in \mathbb{D} \subset \mathbb{R}^r$, and only a local result, with reference to Corollary 3, can be proved. If the actuator/effector mapping takes the form $h(t, x, u) := \Phi(t, x, u)\theta$, where θ is a unknown parameter vector, an adaptive law may be included in the design and, Σ_2 expand. In the case of $\Phi(t, x, u)$ not persistently exited, one need to rely on Corollary 2 in order to conclude stability of the cascade.

Remark 4: It is important to notice that Lemma 1 enables us to use the dynamic optimizing control allocation approach initially formulated in [9] for a wider class of nonlinear systems by relaxing system assumptions directly. For example, the functions f and g in (1) may be assumed locally Lipschitz, instead of globally Lipschitz. Also by relaxing the demands on the subsystem performance (virtual controller and optimal search) from exponential to asymptotic convergence, a more general class of nonlinear systems may be studied.

V. Conclusions and further work

Based on a previous result about uniform global asymptotic stability (UGAS) of the equilibrium of a cascaded time-varying systems a similar result for a set-stable cascaded systems is established. It was also suggested that more general nonlinear systems may be considered for the dynamic optimizing control allocation approach presented in [9] and [10], by using the main result of this note.

The main focus of further work may be to provide and formalize ways of guaranteeing UGB, as in Theorem 1, of the closed loop solutions with respect to the cascaded set, possibly in the framework of [1].

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