ON TRANSIENT DYNAMICS, OFF-EQUILIBRIUM BEHAVIOUR AND IDENTIFICATION IN BLENDED MULTIPLE MODEL STRUCTURES

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Abstract

The use of multiple-model techniques has been reported in a variety of control and signal processing applications. However, several theoretical analyses have recently appeared which outline fundamental limitations of these techniques in certain domains of application. In particular, the identifiability and interpretability of local linear model parameters in transient operating regimes is shown to be limited. Some modifications to the basic paradigm are suggested which overcome a number of the problems. As an alternative to parametric identification of blended multiple model structures, nonparametric Gaussian process priors are suggested as a means of providing local models, and the results compared to a multiple-model approach in a Monte Carlo simulation on some simulated vehicle dynamics data.

1 The Multiple-Model Framework

The past few years have shown an increase in the use of *local model representations* of non-linear dynamic systems (see (Johansen and Murray-Smith 1997) for a review). This basic structure includes a number of approaches: *Tagaki–Sugeno* fuzzy systems (Takagi and Sugeno 1985), local model networks, gain-scheduled control (Leith and Leithead 1999), and statistical mixture models, among them. The model parameters are obtained from prior knowledge, linearisations of a physical model or identified from measured data. Advantages of this approach are purported to be its simplicity, the insight into global dynamics obtained from the local models, and the ease with which global control laws can be constructed from local designs.

Consider the nonlinear system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$. By a blended local model structure we understand a dynamic model of the form

$$\dot{\mathbf{x}} = \sum_{i=1}^{N_m} \rho_i(\mathbf{x}, \mathbf{u}) \mathbf{f_i}(\mathbf{x}, \mathbf{u}), \tag{1}$$

where state $\mathbf{x} \in \mathrm{I\!R}^N$, input $\mathbf{u} \in \mathrm{I\!R}^P$, the model $\mathbf{f_i}(:,:)$ is one of N_m vector functions of the state and the input, and is valid in a region defined by the scalar validity function ρ_i, which in turn is a function of the above variables. Typically, the local models $\mathbf{f}_i$ are chosen to be of the form $\mathbf{f}_i(\mathbf{x}, \mathbf{u}) = \mathbf{A_i x} + \mathbf{B_i u} + \mathbf{d_i}$, resulting in constituent dynamic systems Σ_i given by,

$$\Sigma_i : \dot{\mathbf{x}} = \mathbf{f}_i(\mathbf{x}, \mathbf{u}) = \mathbf{A_i x} + \mathbf{B_i u} + \mathbf{d_i}, \tag{2}$$

where $\mathbf{x}, \mathbf{d}_i \in \mathrm{I\!R}^N$, $\mathbf{A}_i \in \mathrm{I\!R}^{N \times N}$, and $\mathbf{B}_i \in \mathrm{I\!R}^{N \times P}$. This results in a non-linear description of plant dynamics of the form,

$$\dot{\mathbf{x}} = \mathbf{A}(\mathbf{x}, \mathbf{u})\mathbf{x} + \mathbf{B}(\mathbf{x}, \mathbf{u})\mathbf{u} + \mathbf{d}(\mathbf{x}, \mathbf{u}), \tag{3}$$

where $\mathbf{A}(\mathbf{x}, \mathbf{u}) = \sum_i^{N_m} \rho_i(\mathbf{x}, \mathbf{u})\mathbf{A_i}$, $\mathbf{B}(\mathbf{x}, \mathbf{u}) = \sum_i^{N_m} \rho_i(\mathbf{x}, \mathbf{u})\mathbf{B_i}$ and $\mathbf{d}(\mathbf{x}, \mathbf{u}) = \sum_i^{N_m} \rho_i(\mathbf{x}, \mathbf{u})\mathbf{d_i}$.

2 Limitations of the approach

Several limitations of the multiple model approach are reviewed in (Shorten *et al.* 1999). These limitations can be summarised as being either philosophical or technical in nature. By philosophical limitations we mean difficulties, or confusion, in understanding the meaning of the multiple-model model. Such an understanding may be vital when we come to use the model for designing a control system.

In this paper we concentrate on selected technical limitations of the multiple model framework. In particular, we are interested in studying the difficulties associated with modelling off-equilibrium behaviour in dynamic systems. Furthermore, we also focus on the model bias that results from any assumptions on the model structure and data, or from unbiasedness of the identification algorithm. Roughly speaking, these limitations, from a practical perspective, are related to the identifiability and interpretability of the local models. The problem of identifying off-equilibrium linear models stems from the difficulty in gathering data of sufficient quality in these regions. In order to illustrate how interpretability problems arise in multiple model systems we present the following abstract example of model construction using multiple models.

Consider the abstract case of approximating the flow associated with the dynamic system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, in the vicinity of some

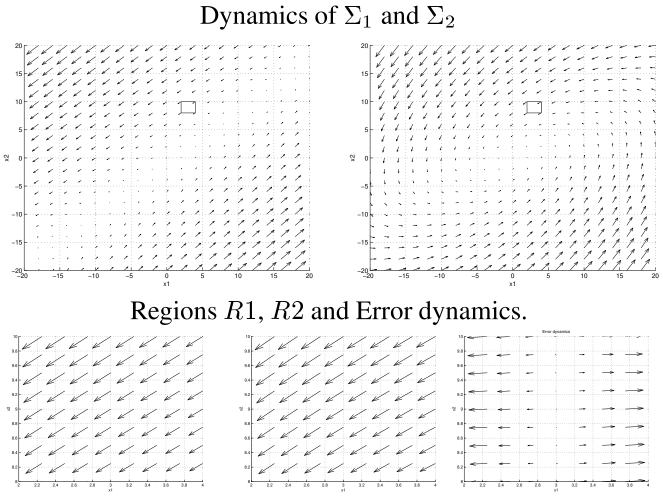

Figure 1: Non-uniqueness of representation. The two systems Σ_1 and Σ_2 are qualitatively different, but in the outlined regions (Shown in detail in $R1$ and $R2$) we see that there is little difference, as shown by the error dynamics.

vector $\mathbf{x_0}$ by the local model $\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{d}$, where $\mathbf{f}(:) \in \mathbb{R}^{\mathbf{N}}$, where $\mathbf{x}, \mathbf{A}, \mathbf{d}$ are as defined earlier. Clearly for any arbitrary choice of invertible $\mathbf{A}$, regardless of its nature (stable, unstable, complex, etc.), a vector $\mathbf{d}$ can be found such that

$$\mathbf{f}(\mathbf{x}_0) = \mathbf{A}\mathbf{x}_0 + \mathbf{d}, \tag{4}$$

where $\mathbf{d} = \mathbf{f}(\mathbf{x}_0) - \mathbf{A}\mathbf{x}_0$. Hence, at $\mathbf{x} = \mathbf{x}_0$ a non-unique parameterisation of the dynamics exist, and indeed the linearisation is meaningless. Furthermore, in the neighbourhood of $\mathbf{x_0}$, subject to some approximation error, by simply varying the location of the virtual equilibria (or the form of the ρ_i), it is possible to obtain many (dynamically) different parameterisations of the non-linear dynamics.[1] This is illustrated in the following example.

Example Consider the behaviour of the following autonomous systems,

$$\begin{aligned} \Sigma_1 : \dot{\mathbf{x}} &= \mathbf{A_1}\mathbf{x}, & (5)\\ \Sigma_2 : \dot{\mathbf{x}} &= \mathbf{A_2}\mathbf{x} + \mathbf{d}, & (6) \end{aligned}$$

where $\mathbf{A_1} = \left[\begin{smallmatrix} 4 & -4.5 \\ 4.05 & -4.55 \end{smallmatrix}\right]$, $\mathbf{A_2} = \left[\begin{smallmatrix} 0.51 & -4.29 \\ 3.84 & -4.51 \end{smallmatrix}\right]$, $\mathbf{d} = \left[\begin{smallmatrix} -8.58 \\ -0.27 \end{smallmatrix}\right]$. The flow associated with both of these systems is depicted in Figure 1. These systems are qualitatively very different; Σ_1 is a *stable node* with an equilibrium point centered at the origin, whereas Σ_2 is a *stable spiral* with its equilibrium point close to, but not centred, at the origin. However, in a small region defined by $R : 2 \le x_1 \le 4,\ 8 \le x_2 \le 10$, as depicted, the flow of both systems is similar. The velocity vectors point in the same direction and the maximum error, defined by,

$$\epsilon_{max} \;=\; \max_{x \in R} \frac{\| (\mathbf{A_2} - \mathbf{A_1})\mathbf{x} + \mathbf{d} \|}{\| \mathbf{x} \|} < 0.42. \tag{7}$$

is bounded and small. The error dynamics $\dot{\mathbf{x}} = (\mathbf{A_2} - \mathbf{A_1})\mathbf{x} + \mathbf{d}, \mathbf{x} \in R$ are depicted in Figure 1. Hence, we conclude that in R, subject to some appropriately defined approximation error, the dynamics described by Σ_1 and Σ_2 are in some sense equivalent. In this region both Σ_1 and Σ_2 are valid representations of an appropriate non-linear system, but outside the region they differ considerably.

This rather obvious observation is of crucial importance for two reasons. First, the identifiability of the local model parameters is poor as a direct consequence of the fact that off-equilibrium local models with singificantly different parameters may give very similar dynamics within their region of validity. Secondly, it strongly suggests that the qualitative nature of the identified local models may say very little about the non-linear dynamics even locally. This is by virtue of the fact that the local model is, by definition, only valid in a local region of state space, and crucially in the off-equilibrium case, that the local model's contribution to the global model only comes from a restricted sub-region which does not include the model's equilibrium point. This observation, in conjunction with many similar observations in control and identification contexts, is referred to as the *Paradox of Locality* in Local Model Networks.

Another problem we want to emphasize is the bias being introduced due to the a priori model structure assumptions. Structure identification and identification of the ρ_i functions might obviously improve on this. However, in some cases the multiple model structure is not ideally suited to the system structure and a significant bias might be difficult to avoid.

3 Revising the Off-Equilibria Multiple Model Framework

It was recently shown in (Johansen *et al.* 1998) that the finite set of linearizations about a finite number of points (equilibria and transient points) can be used to accurately approximate dynamic linearization about arbitrary trajectories using a blended multiple model structure. Despite the theoretical importance of this result, it is clear that the identification problems outlined in Section 2 are paramount in a practical context. In this section we describe two complementary approaches for revising the basic multiple model framework described in Section 2. The first approach involves suggesting modifications to the existing framework which alleviate, to some extent, the problems. The second approach involves the development of a complementary nonparametric framework, with the specific aim of eliminating the problems outlined. The efficacy of both approaches is compared by means of a simple example in Section 4.

3.1 Modification of Existing Framework

The dynamic linearization of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$ about the point $(\mathbf{x}'_0, \mathbf{u}'_0)$ on some arbitrary trajectory is given by

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}'_0, \mathbf{u}'_0) + \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}'_0, \mathbf{u}'_0)(\mathbf{x} - \mathbf{x}'_0) + \frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\mathbf{x}'_0, \mathbf{u}'_0)(\mathbf{u} - \mathbf{u}'_0).$$

[1] We note also that conditions exist such that two systems, which have the same equilibrium point, can be identical along an entire manifold; namely, when $\mathbf{A_1}$ and $\mathbf{A_2}$ share eigenvector and eigenvalue pairs. The manifold is defined by the eigenvectors common to both systems.

Introducing deviation coordinates $\Delta\mathbf{x} = \mathbf{x} - \mathbf{x}_0'$, $\Delta\mathbf{u} = \mathbf{u} - \mathbf{u}_0'$ we get the *small-signal dynamics*

$$\dot{\Delta\mathbf{x}} \;=\; \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}_0', \mathbf{u}_0')\Delta\mathbf{x} + \frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\mathbf{x}_0', \mathbf{u}_0')\Delta\mathbf{u} \qquad (8)$$

that describe the response to small perturbation about a point $(\mathbf{x}_0', \mathbf{u}_0')$ on the nominal trajectory $(\mathbf{x}_0(t), \mathbf{u}_0(t))$. In addition, the *large-signal dynamics* are locally approximated by the equation $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}_0', \mathbf{u}_0')$ which approximates the flow of the state by a constant vector near the point $(\mathbf{x}_0', \mathbf{u}_0')$ along the nominal trajectory $(\mathbf{x_0(t)}, \mathbf{u_0(t)})$. Now suppose we seek local linear models of the form

$$\dot{\mathbf{x}} \;=\; \mathbf{A_i x} + \mathbf{B_i u} + \mathbf{d_i} \qquad (9)$$

to be approximately valid in a neighbourhood of a point $(\mathbf{x_i}, \mathbf{u_i})$. Away from equilibrium, the representation (9) is over-parameterized (non-unique) since only the constant vector term $\mathbf{d_i}$ is sufficient to give an arbitrarily good approximation locally, see also Section 2. The additional degrees of freedom in the parameters $\mathbf{A_i}$ and $\mathbf{B_i}$ can be used in different ways:

- $\mathbf{A_i}$ and $\mathbf{B_i}$ can be selected to increase the region of validity of the local linear approximation (9). In this case these parameters may be completely different from the small-signal model parameters $\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x_i}, \mathbf{u_i})$ and $\frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\mathbf{x_i}, \mathbf{u_i})$ and serve only the purpose of providing a richer class of function approximators. Consequently, the local linear model may not be interpretable in terms of a small/large-signal decomposition.

- $\mathbf{A_i}$ and $\mathbf{B_i}$ can be selected to accurately represent the small-signal dynamics, i.e. $\mathbf{A_i} \approx \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x_i}, \mathbf{u_i})$ and $\mathbf{B_i} \approx \frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\mathbf{x_i}, \mathbf{u_i})$. As a consequence, the offset term will approximately characterize the large-signal dynamics, i.e. $\mathbf{d_i} \approx \mathbf{f}(\mathbf{x_i}, \mathbf{u_i}) - \mathbf{A_i x_i} - \mathbf{B_i u_i}$. This is advantageous in terms of interpretation, analysis and applicability of the model in control systems design, but may have the disadvantage that it may lead to a smaller region of validity of the local model.

Identification of the parameters of (9) using, for example, a standard least-squares criterion and some experimental data will only interpret the local model as an approximator and thus not necessarily lead to local model parameters $\mathbf{A_i}$ and $\mathbf{B_i}$ with a valid small-signal model interpretation. The identifiability problem is amplified by the experience that typically there is very sparse information about small-signal dynamics in transient operating regimes available in the data. The reasons for this are diverse: The system typically spends little time in transient conditions compared to stationary operating conditions, and the large signals components in the data will dominate the identification criterion.

Carefully planned and expensive experiments are required in order to get even a small amount of small-signal dynamics information in transient operating regimes. In order to get data which are informative with respect to both equilibrium and transient local models, the data should consist of two different types of excitation signals: Standard small signal perturbations (e.g. PRBS tests) about the relevant equilibrium points of the system, and high-frequency large step signals with superpositioned large signal perturbations moving the system through the relevant transient states. For the purpose of setpoint control, we often require that the equilibrium local models have significantly higher accuracy than the off-equilibrium ones.

Constrained and regularized identication is in general a useful tool when the data are not sufficiently informative. Robust identification can also be achieved by directly constraining the local model parameters during identification, see e.g. (Johansen 1997). Another possibility is to take advantage of the regularizing effect of locally weighted identification methods where each local model is identified separately by weighting each data sample according to its relevance for the local model (Murray-Smith and Johansen 1997). It is observed that this usually leads to local models with a more valid small-signal interpretation than the standard global identification method. However, since these identification algorithms are biased (Johansen 1997, Murray-Smith and Johansen 1997) compared to the unbiased common global least squares identification algorithm, this improvement will usually be achieved at the cost of a significantly increased model bias with the result that the overall prediction performance of the model may be reduced. It is therefore important that the model structure is well tuned to minimize the bias. Eventually, we are facing the well-known bias/variance tradeoff.

3.2 Nonparametric alternatives

Nonparametric models retain the available data and perform inference conditional on the current state and local data (called 'smoothing' in some frameworks). As the data are used directly in prediction, unlike the parametric methods more commonly used in control contexts, nonparametric methods have advantages for off-equilibrium regions. The uncertainty of model predictions can be made dependent on local data density, and the model complexity automatically related to the amount of available data (more complex models need more evidence to make them likely). Both aspects are very useful in sparsely-populated transient regimes. Moreover, since weaker prior assumptions are typically applied in a nonparametric model, the bias is typically less.

An example of the use of a nonparametric model is a *Gaussian Process prior*, as reviewed in (Williams 1998). In the following, the full matrix of state and control input vectors is denoted Ψ, and the vector of output points is $\mathbf{y}$. The discrete data of the regression model are $\psi_k = [\mathbf{x}(t), \mathbf{u}(t)]$ and $\mathbf{y}_k = \dot{\mathbf{x}}(t)$. The given N_1 data pairs used for identification are stacked in matrices $\Psi_1, \mathbf{y}_1$ and the N_2 data pairs used for prediction are $\Psi_2, \mathbf{y}_2$. Instead of parameterising $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$ as a multiple model, we can place a prior directly on the space of functions where $\mathbf{f}$ is assumed to belong. A Gaussian process represents the simplest form of prior over functions, so for the case with partitioned data $\mathbf{y}_1$ and $\mathbf{y}_2$ we will have the multivariate Normal distribution (we will assume zero mean),

$$\left[\begin{array}{c}\mathbf{y}_1\\ \mathbf{y}_2\end{array}\right] \sim \mathcal{N}\left(0,\Sigma\right), \quad \Sigma = \left[\begin{array}{cc}\Sigma_1 & \Sigma_{12}\\ \Sigma_{21} & \Sigma_2\end{array}\right]. \quad (10)$$

where Σ is the full covariance matrix, and $\Sigma_{21} = \Sigma_{12}^T$. Like the Gaussian distribution, the Gaussian Process is fully specified by a mean and its covariance function. The Normal assumption may seem strangely restrictive initially, but we have a powerful tool in that we can adapt the model's prior expectations to a given application by altering the structure and parameters of the covariance function. The covariance function $C(\psi_i, \psi_j) = \Sigma_{ij}$ (the ij-element of Σ) expresses the expected covariance between y_i and y_j – we can therefore infer $\mathbf{y}_2$'s from constant $(\Psi_1, \mathbf{y}_1)$'s rather than building explicit models. We will also often view the covariance function as being the combination of a covariance function due to the underlying model C_m and one due to measurement noise C_n. The entries of this matrix are then: $\Sigma_{1_{ij}} = C_m(\psi_{1_i}, \psi_{1_j}; \Theta) + C_n(\psi_{1_i}, \psi_{1_j}; \Theta)$, where $C_n()$ could be $\delta_{ij}\sigma_n^2$, for Gaussian noise of variance σ_n. In this paper, we use a straightforward smoothness prior covariance function which states that outputs associated with ψ's closer together should have higher covariance than points further apart,

$$C_m(\psi_i, \psi_j; \Theta) = v_0 \rho(|\psi_i - \psi_j|, \alpha). \quad (11)$$

$\rho(d)$ is a distance measure, which should be one at $d = 0$ and which should be a monotonically decreasing function of d. The one used here was

$$\rho(|\psi_i - \psi_j|, \alpha) = e^{-\frac{1}{2}\sum_{k=1}^{p} \alpha_k (\psi_{ik} - \psi_{jk})^2}. \quad (12)$$

The α_k's determine how quickly the function varies in dimension k. This estimates the relative smoothness of different input dimensions. The parameter vector $\Theta = (v_0, \alpha_{1,..p})$ can be adapted using standard gradient-based optimisation tools.

The choice of covariance function is only constrained in that it must always generate a non-negative definite covariance matrix for any Ψ, so we can represent a spectrum of systems from very local nonlinear models, to standard linear models using the same framework.

As in the multinormal case, we can divide the joint probability (10) into a marginal Gaussian process and a conditional Gaussian process

$$p(Y) = p(\mathbf{y}_1, \mathbf{y}_2) = p(\mathbf{y}_1)p(\mathbf{y}_2|\mathbf{y}_1). \quad (13)$$

The marginal term gives us the likelihood of the training data,

$$P(\mathbf{y}_1) = (2\pi)^{-\frac{N_1}{2}} |\Sigma_1|^{-\frac{1}{2}} e^{-\frac{1}{2}\mathbf{y}_1^T \Sigma_1^{-1} \mathbf{y}_1}. \quad (14)$$

The conditional part of the model, which best relates to a traditional regression model is therefore the Gaussian process which gives us the output p.d.f. conditional on the training data $\Psi_1, \mathbf{y}_1$ and the test points Ψ_2.

$$\begin{aligned} P(\mathbf{y}_2|\mathbf{y}_1) &= \frac{P(\mathbf{y}_2, \mathbf{y}_1)}{P(\mathbf{y}_1)} && (15)\\ &= \frac{e^{-\frac{1}{2}(\mathbf{y}_2 - \mu_{21})^T \Sigma_{2.1}^{-1} (\mathbf{y}_2 - \mu_{21})}}{(2\pi)^{\frac{N_2}{2}} |\Sigma_{21}|^{\frac{1}{2}}}, && (16)\end{aligned}$$

where, as in the straightforward multinormal case,

$$\begin{aligned} \mu_{21} &= \Sigma_{12}^T \Sigma_1^{-1} \mathbf{y}_1 && (17)\\ \Sigma_{21} &= \Sigma_2 - \Sigma_{12}^T \Sigma_1^{-1} \Sigma_{21}, && (18)\end{aligned}$$

so, as Σ is dependent on Ψ we can view this as a nonlinear regression and use $f(\Psi_2) = \mu_{21}$ as the expected model output, with a variance of $\sigma^2(\Psi_2) = \Sigma_{21}$.

One advantage of the Gaussian process is that, for differentiable covariance functions, it is easy to produce analytic linearisations (in a limit in mean square sense) of the model's mean prediction (which are also Gaussian processes).[2]

4 Vehicle dynamics example

As an example, consider the longitudinal dynamics of a vehicle with mass m and speed v. The interesting aspect of this experiment, over and above its practical relevance, is that it is a 1st order system, the nonlinearity is fairly smooth, and we are using noise-free data, but as we will see, identification of the nonlinear model can be surprisingly difficult. The vehicle is powered by an engine which generates a longitudinal force $g_e(v, u)$ where u is the throttle angle. The vehicle is subject to a disturbance force g_d. A simple first order model of the vehicle is given by the force balance $m\dot{v} = g_e(v, u) - g_d$, which can be written

$$\dot{v} \;=\; f(v, u) \;=\; (g_e(v, u) - g_d)/m. \quad (19)$$

In the example, we set $g_d = 1000$ N, $m = 1000$ kg and the engine characteristic,

$$g_e(v, u) = (1 + 3u)(1 + \arctan(6u^2 - 0.4u + 1.2)) \cdot 500\ N,$$

is shown in Figure 2. With this characteristic engine curve (which corresponds to a fixed gear ratio), the engine operates in a speed interval between 2 and 20 m/s.[3] Linearization of the engine model (19) leads to the small-signal parameters $\frac{\partial f}{\partial v}(v, u)$ and $\frac{\partial f}{\partial u}(v, u)$ and drift term $f(v, u)$. These parameters are illustrated in Figure 3. The experimental data used for identification were obtained by excitation of the vehicle by an input signal containing both large and small amplitude changes in order to determine the large-signal and small-signal parts of the off-equilibrium local models.

To evaluate the model performance and robustness we generated 100000 data points, and identified 40 models on nonoverlapping subsets of 700 points. In each test the models were then tested on the remaining 99300 points. The prediction performance of the different models are summarised in Table 1.

[2] We can thus also analytically derive the variance of the derivative mean – an aspect of Gaussian Process priors which is difficult to reproduce in other models without extensive simulation.

[3] This example is motivated by the experimental vehicle speed control problem considered in (Johansen *et al.* 1998). The model is simplified, but contains the relevant aspects of the experimental vehicle in order to illustrate the main ideas.

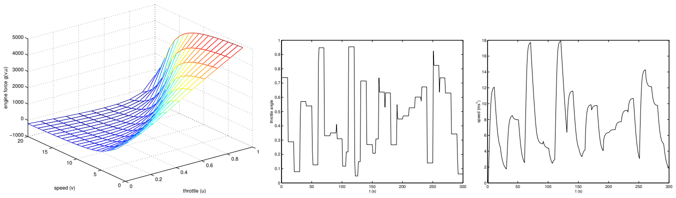

Figure 2: Left: Engine force. Right hand plots are time series of a subsequence with 300 data points.

Figure 3 gives insight into the identified parameters of the multiple model, using locally weighted identification of model parameters, and Gaussian Process approaches, using a fixed covariance function.

We observe that the Gaussian Process approach produces more accurate estimates of the small-signal parameters than the multiple model approach with locally weighted regression. It can be seen that the model bias is the main contribution to errors in the estimated small-signal parameters with the multiple model approach. With the Gaussian process approach the bias dominates along the equilibrium manifold, while the variance becomes more significant far away from equilbrium where data are sparse. Using global least squares, the prediction performance of the blended multi-model can be improved, mainly due to reduced bias because it is an unbiased identification algorithm, cf. Table 1. In this example we found it difficult to reduce the bias of the multiple model structure without decreasing the overall accuracy due to increased variance. Note that with a less favourable experiment design we have experienced that the differences between multiple models and Gaussian processes become even more pronounced. With the current experiment design and because we are dealing with a first order system, the off-equilbrium local models can be identified fairly well, and we do not experience high variability of these local model parameters.

Although the Gaussian Process has consistently better test results from the given data, and in the region covered by data the variance is evenly low (unlike the multiple model results which increase evenly with distance from the equilibria), but as we move away from that to the edge of the plots we see a great increase in variance of the derivative means. This is however, unlike with parametric methods, accompanied by a related increase in expected prediction variance (analytic variance estimates for the GP's (not plotted) grew accurately in sparsely populated areas, as desired, and match well with the Monte Carlo results shown). Note also the large levels of bias in the multiple model plots, which somewhat skew the test results in the favour of the Gaussian Process.

5 Conclusions

We have outlined theoretical problems with the multiple model framework when representing off-equilibrium behaviour, and illustrated them in Monte Carlo simulations. Consideration of these problems leads to new approaches to experiment planning and more robust identification methods when optimising local model parameters. This should also provide us with more interpretable models suitable for subsequent control system design.

An alternative approach, based on nonparametric *Gaussian Process prior* models was developed and found to provide an interesting extension of the multiple model framework, which is simple and elegant, and can model nonlinear problems in a probabilistic framework. The disadvantage is its computational complexity, as prediction of model outputs requires a matrix inversion of the $N_1 \times N_1$ covariance matrix Σ_1, which becomes problematic for identification data where $N_1 >> 1000$. In transient regimes, however, one typically has very few data points and we wish to make robust estimates of model behaviour. This suggests a heterogeneous solution with a multiple-model model composed of a number of linear submodels around equilibrium points, and Gaussian process submodels in transient areas.

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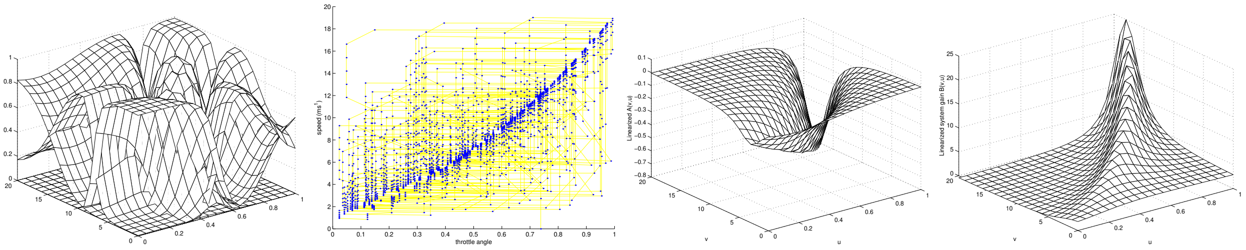
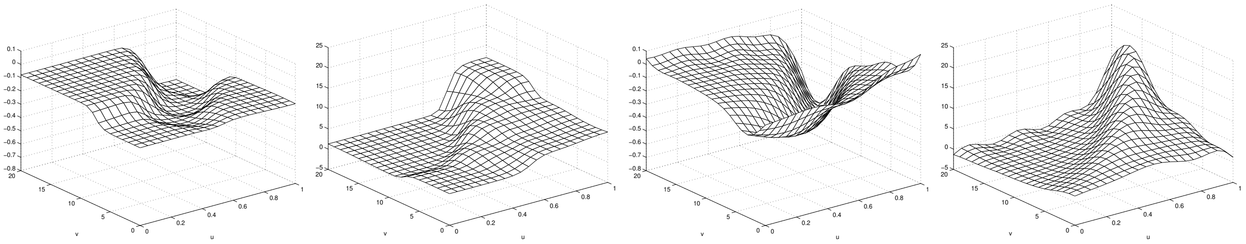
Table 1: The m.s.e. is the root mean square error when predicting $\dot{v}$ based on u and v, σ_e is standard deviation of this error.

	Full set		Off-equil.		On-equil.	
Model	m.s.e.	σ_e	m.s.e.	σ_e	m.s.e.	σ_e
Multiple models (locally weighted fit)	0.1034	0.00497	0.1387	0.00855	0.0798	0.00890
Multiple models (global fit)	0.0699	0.00389	0.0874	0.00525	0.0587	0.00844
Gaussian Process	0.0057	0.00326	0.0057	0.00332	0.0057	0.00325

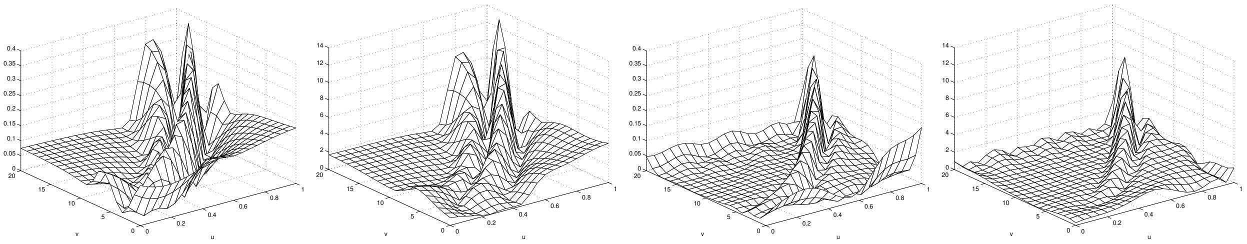
From left: Multiple model weighting functions ρ_i, example phase plot of data (speed vs. throttle angle). Right hand plots are exact system linearisations: $\frac{\partial f}{\partial v}(v, u)$, $\frac{\partial f}{\partial u}(v, u)$.



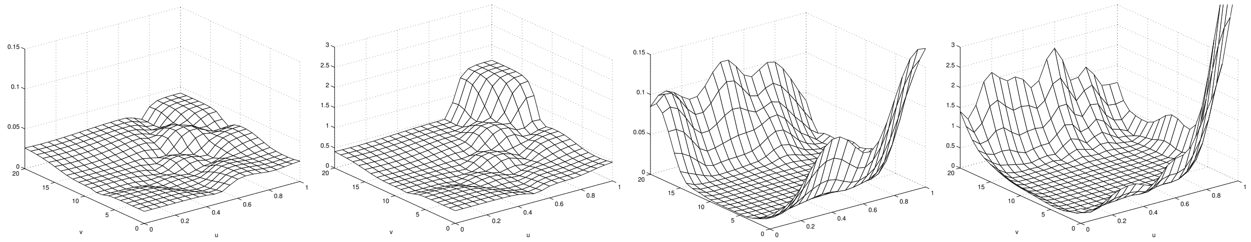
Mean multiple model small-signal parameters $A(v, u)$ and $B(v, u)$, mean GP estimates of small-signal parameters $\frac{\partial f}{\partial v}(v, u)$, $\frac{\partial f}{\partial u}(v, u)$.



Absolute bias in multiple model small signal-parameters $A(v, u)$ and $B(v, u)$ (left two) and GP small-signal parameters $\frac{\partial f}{\partial v}(v, u)$, $\frac{\partial f}{\partial u}(v, u)$ (right two).



From left: 2 times the standard deviation of the multiple model small signal-parameters $A(v, u)$ and $B(v, u)$ (left two) and GP small-signal parameters $\frac{\partial f}{\partial v}(v, u)$, $\frac{\partial f}{\partial u}(v, u)$ (right two).



From left: Mean bias and 2 std. dev. plots for $f(v, u)$, for multiple model (left two) and GP (right two)

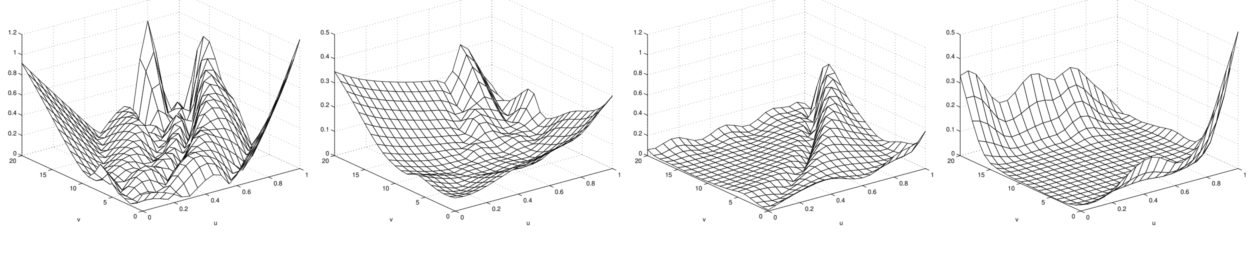


Figure 3: Results of Monte Carlo simulation of multiple model and Gaussian Process identification processes. Note that we plot the means of 40 models here, which are usually better models than the individual models.