Adaptive Control Allocation

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Abstract

In this work we address the control allocation problem for a nonlinear over-actuated time-varying system where parameters affine in the effector model may be assumed unknown. Instead of optimizing the control allocation at each time instant, a dynamic approach is considered by constructing update-laws that represent asymptotically optimal allocation search and adaptation. Using Lyapunov analysis for cascaded set-stable systems, uniform global/local asymptotic stability is guaranteed for the sets described by the system, the optimal allocation update-law and the adaptive update-law.

Key words: Control allocation; Adaptive control; Nonlinear systems; Cascaded systems; Optimization.

1 Introduction

This work is motivated by the over-actuated control allocation problem. The problem is described by a nonlinear system, divided into a dynamic and a static part. The main contribution of this work is to show that the instantaneous control allocation problem, optimizing the desired input of the dynamic part of the system, not necessarily needs to be solved exactly at each time instant. In order to imply convergence and stability properties of the whole system, stability and attractivity of the optimal set, described by the control allocation problem, is pursued.

Optimizing control allocation solutions have been derived for certain classes of over-actuated systems, such as aircraft, automotive vehicles and marine vessels, (Enns 1998, Buffington, Enns & Teel 1998, Sjørdalen 1997, Bodson 2002, Härkegård 2002, Luo, Serrani, Yurkovich, Doman & Oppenheimer 2004, Luo, Serrani, Yurkovich, Doman & Oppenheimer 2005, Poonamallee, Yurkovich, Serrani, Doman & Oppenheimer 2005, Johansen, Fossen & Berge 2004) and (Johansen, Fossen & Tøndel 2005). The control allocation problem is, in (Enns 1998, Buffington et al. 1998, Sjørdalen 1997, Johansen et al. 2005, Bodson 2002) and (Härkegård 2002), viewed as a *static* or *quasi-dynamic* optimization problem that is solved independently of the dynamic control problem considering non-adaptive linear effector models of the form $\tau = Gu$. In (Luo et al. 2004) and (Luo et al. 2005) a dynamic model predictive approach is considered to solve the allocation problem with linear time-varying dynamic in the actuator/effector model, $T\dot{u} + u = u_c$. In (Poonamallee et al. 2005) and (Johansen et al. 2004) sequential quadratic programming tech-

niques are used to cope with nonlinearities in the control allocation problem due to singularity avoidance and actuator failure. The main advantage of the control allocation approach is in general modularity and the ability to handle redundancy and constraints.

Consider the system

$$\dot{x} = f_x(t, x) + g_x(t, x)\tau \quad (1)$$

$$\tau = \Phi(t, x, u, \theta) := \Phi_\theta(t, x, u)\theta + \Phi_0(t, x, u) \quad (2)$$

where $t \geq 0$, $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^r$ is control input vector, $\tau \in \mathbb{R}^d$ is a vector of virtual controls and $\theta \in \mathbb{R}^m$ is a constant vector that contains parameters that either will be assumed known, or viewed as uncertain parameters to be adapted. The state x , is assumed known and typically represents the tracking error. The mapping (2) represents an actuator and effector model, where the control signals u , are mapped to the virtual controls τ . We limit our study to the over-actuated control allocation problem, where $m \leq d \leq r$.

Problem: Assume that there exist a virtual control $\tau_c := k(t, x)$, with $k(t, 0) = 0$, that is differentiable and uniformly globally asymptotically stabilizes the origin of (1), when $\tau = k(t, x)$.

- i) If θ is known, define an update law $\dot{u} := f_u(t, x, u)$, such that the stability property of the closed loop is conserved and $u(t)$ converges to an optimal solution with respect to the static minimization problem

$$\min_u J(t, x, u) \quad \text{s.t.} \quad \tau_c - \Phi(t, x, u, \theta) = 0. \quad (3)$$

- ii) If θ is unknown, solve problem i) with $\theta = \hat{\theta}$, $\dot{u} := f_u(t, x, u, \hat{\theta})$ and define an adaptive update law $\dot{\hat{\theta}} := f_{\hat{\theta}}(t, x, u, \hat{\theta})$, where $\hat{\theta} \in \mathbb{R}^m$ is an estimate of θ .

In (Johansen 2004) it was shown that it is not necessary to solve the optimization problem (3) exactly at each time instant. Further a control Lyapunov function was used to derive an exponentially convergent update-law for u (similar to a gradient or Newton-like optimization) such that the control allocation problem (3) could be solved dynamically. And finally it was shown that convergence and asymptotic optimality of the system, composed by the dynamic control allocation and a uniform globally exponentially stable trajectory-tracking controller, guarantees uniform boundedness and uniform global exponential convergence to the optimal solution of the system. The advantage of this approach is computational efficiency and simplicity of implementation, since the optimizing control allocation algorithm is implemented as a dynamic nonlinear controller. Solving (3) online at each sampling instant requires a computationally more expensive numerical solution. In (Tjønnås & Johansen 2005) the results from (Johansen 2004) were extended by allowing uncertain parameters, associated with an adaptive law, in the effector model, and by applying set-stability analysis in order to also conclude asymptotic stability of the optimal solution set.

In what follows we will extend the ideas from (Johansen 2004) and (Tjønnås & Johansen 2005) by utilizing the set-stability result for cascaded systems established in (Tjønnås, Chaillet, Panteley & Johansen 2006). This result enables us to relax the assumptions in (Johansen 2004) and (Tjønnås & Johansen 2005) where $f_x(t, x)$, $g_x(t, x)$ and $\Phi(t, x, u, \theta)$ are assumed to be globally Lipschitz in x . Further the virtual controller τ_c does only need to render equilibrium of (1) UGAS, not UGES as assumed in (Johansen 2004) and (Tjønnås & Johansen 2005). The implementation of the adaptive law presented in (Tjønnås & Johansen 2005) depends directly on the Lyapunov function used in the analysis. In this work the analysis and implementation are separated, and although the Lyapunov functions in the analysis need to satisfy certain requirements, the adaptive implementation does not assume knowledge of these Lyapunov functions.

2 Notation, definitions and preliminary results

$|\cdot|$ denotes the Euclidian norm and $|\cdot|_{\mathcal{A}'} : \mathbb{R}^q \rightarrow \mathbb{R}_{\geq 0}$ denotes the distance from a point $z \in \mathbb{R}^q$ to a set $\mathcal{A}' \subset \mathbb{R}^q$, $|z|_{\mathcal{A}'} := \inf \{|z - y| : y \in \mathcal{A}'\}$. The solution of an autonomous dynamic system $\dot{z} = F(z)$ is denoted by $z(t, z_0)$ at $t \geq t_0$, where $z_0 = z(t_0, z_0)$ is the initial state and t_0 is the initial time.

2.1 Definitions

The definitions that follows are either motivated by, or can be found in (Teel, Panteley & Loria 2002) and (Lin, Sontag & Wang 1996). They pertain to systems of the form

$$\dot{z} = F(z), \quad (4)$$

where $z := (p, x^T)^T$, p is the time-state

$$\dot{p} = 1, \quad p_0 = t_0. \quad (5)$$

and $F : \mathbb{D} \rightarrow \mathbb{R}^q$ is locally Lipschitz with $\mathbb{D} \subset \mathbb{R}^q$. In the following, if referred to a set, it has the properties of being *nonempty*.

Definition 1 The system (4) is said to be forward complete if, for each $z_0 \in \mathbb{D}$, the solution $z(\cdot, z_0) \in \mathbb{D}$ is defined on $\mathbb{R}_{\geq t_0}$.

Definition 2 The system (4) is said to be finite escape time detectable through $|\cdot|_{\mathcal{A}}$, if any solution, $z(t, z_0) \in \mathbb{D}$, which is right maximally defined on a bounded interval $[t_0, T)$, satisfies $\lim_{t \nearrow T} |z(t, z_0)|_{\mathcal{A}} = \infty$.

Definition 3 If the system (4) is forward complete, then the closed set $\mathcal{A} \subset \mathbb{D}$ is:

- Uniformly Stable (US), if there exists a function, $\nu \in \mathcal{K}$, and a constant, $c > 0$, such that, $\forall |z_0|_{\mathcal{A}} < c$,

$$|z(t, z_0)|_{\mathcal{A}} \leq \nu(|z_0|_{\mathcal{A}}), \quad \forall t \geq 0. \quad (6)$$

- Uniformly Globally Stable (UGS), when $\mathbb{D} = \mathbb{R}^q$, if (6) is satisfied with, $\nu \in \mathcal{K}_{\infty}$, and for any $z_0 \in \mathbb{R}^q$.
- Uniformly Attractive (UA) if there exists a constant $c > 0$ such that for all $|z_0|_{\mathcal{A}} < c$ and any $\mu > 0$ there exists $T = T(\mu) > 0$, such that

$$|z_0|_{\mathcal{A}} < c, \quad t \geq T \quad \Rightarrow \quad |z(t, z_0)|_{\mathcal{A}} \leq \mu, \quad (7)$$

- Uniformly Globally Attractive (UGA), when $\mathbb{D} = \mathbb{R}^q$, if for each pair of strictly positive numbers (c, μ) there exists $T = T(\mu) > 0$ such that for all $z_0 \in \mathbb{R}^q$, (7) holds.
- Uniformly Asymptotically Stable (UAS) if it is US and UA.
- Uniformly Globally Asymptotically Stable (UGAS), when $\mathbb{D} = \mathbb{R}^q$, if it is UGS and UGA.

From (Khalil 1996), we adapt the definition of uniform boundedness of solutions to the case when $\mathcal{A}$ is not reduced to $\{0\}$.

Definition 4 With respect to the closed set $\mathcal{A} \subset \mathbb{R}^q$, the solutions of system (4) are said to be:

- Uniformly Bounded (UB) if there exist a positive constant c , such that for every positive constant $r < c$ there is a positive constant $\mu = \mu(r)$, such that

$$|z_0|_{\mathcal{A}} \leq r \quad \Rightarrow \quad |z(t, z_0)|_{\mathcal{A}} \leq \mu, \quad \forall t \geq 0. \quad (8)$$

- Uniformly Globally Bounded (UGB), when $\mathbb{D} = \mathbb{R}^q$, if for every $r \in \mathbb{R}_{\geq 0}$, there is a positive constant $\mu = \mu(r)$ such that (8) is satisfied.

Definition 5 A smooth Lyapunov function for (4) with respect to a non-empty, closed forward invariant set $\mathcal{A} \subset \mathbb{D}$ is a function $V : \mathbb{D} \rightarrow \mathbb{R}$ that satisfies: i) there exists two $\mathcal{K}$ functions α_1 and α_2 such that for any $z \in \mathbb{D}$, $\alpha_1(|z|_{\mathcal{A}}) \leq V(z) \leq \alpha_2(|z|_{\mathcal{A}})$. ii) There exists a continuous and a positive definite or semidefinite function α_3 such that for any $z \in \mathbb{D} \setminus \mathcal{A}$: $\frac{\partial V}{\partial z}(z)F(z) \leq -\alpha_3(|z|_{\mathcal{A}})$.

2.2 Preliminary results

In order to conclude stability of a set of points, rather than a single equilibrium point, we have the following result from (Skjetne 2005).

Theorem 1 Assume that the system (4) is finite escape-time detectable through $z|_{\mathcal{A}}$. If there exists a smooth Lyapunov function for the system (4) with respect to a non-empty, closed, forward invariant set $\mathcal{A}$, then $\mathcal{A}$ is UGS with respect to (4). Furthermore, if α_3 is a positive definite function, then $\mathcal{A}$ is UGAS with respect to (4), and if $\alpha_i(|z|_{\mathcal{A}}) = c_i |z|_{\mathcal{A}}^r$ for $i = 1, 2, 3$ where c_i and r are strictly positive values and $r > 1$, then $\mathcal{A}$ is UGES with respect to (4).

Consider the cascaded system

$$\dot{x}_1 = f_{c1}(t, x_1) + g_c(t, x_1, x_2) \quad (9)$$

$$\dot{x}_2 = f_{c2}(t, x_2). \quad (10)$$

When the functions f_{c1} , f_{c2} and g_c are locally Lipschitz in all arguments, this class of nonlinear time-varying systems can be represented by the following autonomous system

$$\dot{z}_1 = F_1(z_1) + G(z) \quad (11)$$

$$\dot{z}_2 = F_2(z_2) \quad (12)$$

where $z_1 := (p, x_1^T)^T \in \mathbb{R}^{q_1}$, $z_2 := (p, x_2^T)^T \in \mathbb{R}^{q_2}$, $z := (p, x^T)^T \in \mathbb{R}^q$, $x := (x_1^T, x_2^T)^T$, $F_1(z_1) := (1, f_{c1}(p, x_1)^T)^T$, $G(z) := (0, g_c(p, x)^T)^T$ and $F_2(z_2) := (1, f_{c2}(p, x_2)^T)^T$. Based on the cascaded system formulation we will make use of the results from (Tjønnås et al. 2006).

Lemma 1 Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be some closed subsets of $\mathbb{R}^{q_1}$ and $\mathbb{R}^{q_2}$ respectively. Then, under the following assumptions, the set $\mathcal{A} := \mathcal{O}_1 \times \mathcal{O}_2$ is UGAS with respect to the cascade (11)-(12).

A 1 The set $\mathcal{O}_2$ is UGAS with respect to the system (12) and that the solution of the system (11)-(12) is UGB with respect to $\mathcal{A}$.

A 2 The functions F_1 , F_2 and G are locally Lipschitz.

A 3 The cascade (11)-(12) is forward complete.

A 4 There exist a continuous function $G_1 : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ and a class $\mathcal{K}$ function G_2 such that, for all $z \in \mathbb{R}^q$,

$$|G(z)| \leq G_1(|z|_{\mathcal{A}})G_2(|z_2|_{\mathcal{O}_2}). \quad (13)$$

A 5 There exists a continuously differentiable function $\bar{V}_1 : \mathbb{R}^{q_1} \rightarrow \mathbb{R}_{\geq 0}$, class $\mathcal{K}_{\infty}$ functions $\bar{\alpha}_1$, $\bar{\alpha}_2$ and $\bar{\alpha}_3$, and a continuous function $\bar{\varsigma}_1 : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ such that, for all $z_1 \in \mathbb{R}^{q_1}$,

$$\bar{\alpha}_1(|z_1|_{\mathcal{O}_1}) \leq \bar{V}_1(z_1) \leq \bar{\alpha}_2(|z_1|_{\mathcal{O}_1}) \quad (14)$$

$$\frac{\partial \bar{V}_1}{\partial z_1}(z_1)F_1(z_1) \leq -\bar{\alpha}_3(|z_1|_{\mathcal{O}_1}) \quad (15)$$

$$\left| \frac{\partial \bar{V}_1}{\partial x_1}(z_1) \right| \leq \bar{\varsigma}_1(|z_1|_{\mathcal{O}_1}). \quad (16)$$

Corollary 1 Let $\mathcal{O}_1$ and $\mathcal{O}_2 \subset \mathbb{D}$ be closed subsets of $\mathbb{R}^{q_1}$ and $\mathbb{R}^{q_2}$ respectively, and the assumptions A 2 - A 5 be satisfied. Then, with respect to the cascade (11)-(12), the set $\mathcal{A}$ is:

- UGS, when $\mathbb{D} = \mathbb{R}^{q_2}$, if $\mathcal{O}_2$ is UGS with respect to the system (12) and that the solution of system (11)-(12) is UGB with respect to $\mathcal{A}$.
- UAS, if $\mathcal{O}_2$ is UAS with respect to the system (12).
- US, if $\mathcal{O}_2$ is US with respect to the system (12).

3 Optimizing adaptive control allocation

The adaptive control allocation approach is based on three modular steps, see also Figure 1.

- (1) **The high level control algorithm.** The virtual control τ is treated as an available input to system (1), and a virtual control law $\tau_c := k(t, x)$ is designed such that the origin of (1) is UGAS.
- (2) **The control allocation algorithm.** Based on the minimization problem (3) where J is a cost function that incorporates objectives such as minimum power consumption and actuator constraints (implemented as barrier functions), the Lagrangian function
$$L(t, x, u, \lambda) = J(t, x, u) + (k(t, x) - \Phi(t, x, u, \theta))^T \lambda \quad (17)$$
is introduced. Update laws for the control input u and the Lagrangian parameter λ are then defined such that u and λ converges to a set defined by the time-varying optimality condition.
- (3) **The adaptive algorithm.** In order to cope with an unknown parameter vector θ in the effector model, an adaptive law is defined. The parameter estimate is used in the control allocation algorithm and a certainty equivalent adaptive optimal control allocation can be defined.

Fig. 1. Adaptive control allocation design philosophy

3.1 High level control algorithm

In this section we state the main system assumptions and clarify the requirements on the high level control algorithm.

Assumption 1 (Plant)

- a)** There exists a continuous function $G_f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{> 0}$ and a constant K , such that $|f_x(t, x)| \leq G_f(|x|)$ and $|g_x(t, x)| \leq K$ for all t and x . Moreover f_x is locally Lipschitz in t and x , and $f_x(t, 0) = 0$.
- b)** There exist a continuous function $\varsigma_{\partial g_x} : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ such that for all t and x , g_x is differentiable and $\left| \frac{\partial g_x}{\partial t} \right| + \left| \frac{\partial g_x}{\partial x} \right| \leq \varsigma_{\partial g_x}(|x|)$.
- c)** The function Φ is twice differentiable and there exist a continuous function $G_\Phi : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$, such that $|\Phi_\theta| + |\Phi_0| \leq G_\Phi \left(\left| (x^T, u^T)^T \right| \right)$ for all t , x , and u . Further there exist a continuous function $\varsigma_{\partial \Phi}(|x|) \geq \left| \frac{\partial \Phi_\theta}{\partial t} \right| + \left| \frac{\partial \Phi_0}{\partial t} \right| + \left| \frac{\partial \Phi_\theta}{\partial x} \right| + \left| \frac{\partial \Phi_0}{\partial x} \right| + \left| \frac{\partial \Phi_\theta}{\partial u} \right| + \left| \frac{\partial \Phi_0}{\partial u} \right|$ where $\varsigma_{\partial \Phi} : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$.
- d)** There exists constants $\varrho_2 > \varrho_1 > 0$, such that $\forall t, x, u$ and θ

$$\varrho_1 I < \frac{\partial \Phi}{\partial u}(t, x, u, \theta) \left(\frac{\partial \Phi}{\partial u}(t, x, u, \theta) \right)^T \leq \varrho_2 I. \quad (18)$$

Remark 1 Assumption 1 d) can be viewed as a controllability assumption in the sense that: i) the mapping $\Phi(t, x, \cdot, \theta) : \mathbb{R}^r \rightarrow \mathbb{R}^d$ is surjective for all t, x and θ and ii) for all t, x and θ there exist a continuous function $f_u(t, x, \theta)$ such that $\Phi(t, x, f_u(t, x, \theta), \theta) = k(t, x)$. The surjective property can be seen by the Moore-Penrose pseudoinverse solution, $u_1 = \frac{\partial \Phi}{\partial u}|_{u_c}^T \left(\frac{\partial \Phi}{\partial u}|_{u_c} \frac{\partial \Phi}{\partial u}|_{u_c}^T \right)^{-1} y$, of the equation, $y = \frac{\partial \Phi}{\partial u}|_{u_c} u_1$, which exist for any u_c . Thus for every y there exists a solution $\Phi(t, x, u_1, \theta) = \Phi(t, x, 0, \theta) + y$ by the Mean Value Theorem, where $u_c \in (0, u_1)$. ii) follows from the Implicit Function Theorem by i) and Assumption 1 d).

Assumption 2 (High Level Control Algorithm)

- a)** There exists a feedback control $\tau_c := k(t, x) \leq \varsigma_k(|x|)$, where $\varsigma_k : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0}$ is a continuous function, that render the equilibrium of (1) UGAS for $\tau = \tau_c$. The function k is differentiable, $\left| \frac{\partial k}{\partial t} \right| + \left| \frac{\partial k}{\partial x} \right| \leq \varsigma_{\partial k}(|x|)$, where $\varsigma_{\partial k} : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0}$ is continuous, and $k(t, 0) = 0$ for all t .

If we rewrite (1) such that

$$\dot{x} = f(t, x) + g(t, x)(k(t, x) - \Phi(t, x, u, \theta)) \quad (19)$$

where $f(t, x) := f_x(t, x) + g_x(t, x)k(t, x)$, $g(t, x) := -g_x(t, x)$, by Assumption 1 and 2 the functions f and g have the same properties as f_x and g_x . Also the equilibrium of

$$\dot{x} = f(t, x), \quad (20)$$

the set

$$\mathcal{O}_x := \{x \in \mathbb{R}^n | x = 0\}, \quad (21)$$

is UGAS by Assumption 2.

From Assumption 2 a) there exists a Lyapunov function $V_x : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ and $\mathcal{K}_\infty$ functions α_{x1} , α_{x2} , α_{x3} and α_{x4} such that

$$\alpha_{x1}(|x|) \leq V_x(t, x) \leq \alpha_{x2}(|x|) \quad (22)$$

$$\frac{\partial V_x}{\partial t} + \frac{\partial V_x}{\partial x} f(t, x) \leq -\alpha_{x3}(|x|) \quad (23)$$

$$\left| \frac{\partial V_x}{\partial x} \right| \leq \alpha_{x4}(|x|) \quad (24)$$

for the system $\dot{x} = f(t, x)$ with respect to its origin.

Assumption 2 (Continued)

- b)** There exists a $\mathcal{K}_\infty$ function $\alpha_k : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$, such that

$$\alpha_{x4}(|x|)\alpha_k(|x|) \leq \alpha_{x3}(|x|) \quad (25)$$

Remark 2 If the origin of $\dot{x} = f(t, x)$ is UGES, then Assumption 2 is generally satisfied, with for example α_{x3} quadratic, α_{x4} linear and α_k sublinear, and V_x does not need to be known explicitly in order to verify Assumption 2 b).

3.2 Certainty equivalent adaptive control allocation algorithm

In this section we first establish update-laws for the control input u , the Lagrangian multipliers λ , and the parameter estimate $\hat{\theta}$, such that stability and convergence with respect to the time-varying first order optimal set defined by the optimization problem (3) can be concluded. Then we show that the stabilizing properties of the virtual controller from Assumption 2, are conserved for the closed loop system.

In order to account for the unknown, but bounded, parameter vector θ , an adaptive mechanism is include in the optimal control allocation design. We use a Lyapunov based indirect parameter estimation scheme (see for example (Krstic, Kanellakopoulos & Kokotovic 1995) for a systematic Lyapunov based procedure). Since the state vector of (1) is assumed to be known, we consider the estimation model:

$$\dot{\hat{x}} = A(x - \hat{x}) + f_x(t, x) + g_x(t, x)\Phi(t, x, u, \hat{\theta}) \quad (26)$$

in the construction of the adaptive law. This estimation model has the same structure as a typical *series parallel model* (SP) (Ioannou & Sun 1996), (Landau 1979). For analytical purpose, the filtered error estimate of the unknown parameter vector:

$$\dot{\epsilon} = -A\epsilon + g_x(t, x)\Phi_\theta(t, x, u)\tilde{\theta}, \quad (27)$$

where $\tilde{\theta} = \theta - \hat{\theta}$, $\epsilon = x - \hat{x}$ and $(-A)$ is Hurwitz, will be used.

The analysis and design of the certainty equivalent adaptive control allocation algorithm can be carried out in several ways. We consider the approach, with reference to Figure 2, where the perturbing system (Σ_2) is defined by the adaptive law and the allocation algorithm dynamics, while the perturbed system (Σ_1) is defined by the system dynamics. An advantage with this approach

Cascaded System Interpretation

Fig. 2. General representation of a time-variant cascade, where Σ_1 is the *perturbed system* that will be UGAS with respect to a set, $\mathcal{O}_1$, when $|z_2|_{\mathcal{O}_2} = 0$. Σ_2 is the *perturbing system*. Note that Σ_2 may be perturbed indirectly by Σ_1 since z_1 may be considered as a time-varying signal, $z_1(t)$, as long as this signal exists for all t .

is that, the adaptive law is independent of V_x .

Remark 3 A different approach would be to expand the perturbed system (Σ_1) with an adaptive law. In this case the adaptive law will be dependent on the initial Lyapunov function (similar to (Tjonnås & Johansen 2005)), but convergence results like, $x(t) \rightarrow 0$ as $t \rightarrow \infty$, may be concluded even if a persistence of excitation condition is not satisfied.

3.2.1 Stability of the optimal certainty equivalent control allocation

In order to see the cascaded coupling between the system (19) and the adaptive and optimal control allocation update-laws, equation (19) can be rewritten by:

$$\dot{x} = f(t, x) + g(t, x) \left(k(t, x) - \Phi(t, x, u, \hat{\theta}) \right) + g_x(t, x) \Phi_\theta(t, x, u) \tilde{\theta}. \quad (28)$$

Based on the perturbing system, we consider the estimated optimal solution set:

$$\mathcal{O}_{u\lambda\tilde{\theta}}(t, x) := \left\{ (u, \lambda, \epsilon, \tilde{\theta}) \in \mathbb{R}^{n_{u\lambda\tilde{\theta}}} \mid f_{\mathcal{O}_{u\lambda\tilde{\theta}}}(t, z_{xu\lambda\tilde{\theta}}) = 0 \right\} \quad (29)$$

where $n_{u\lambda\tilde{\theta}} := r + d + n + m$, $z_{xu\lambda\tilde{\theta}} := \left(x, u, \lambda, \epsilon, \tilde{\theta} \right)^T$, $f_{\mathcal{O}_{u\lambda\tilde{\theta}}} := \left(\left(\frac{\partial L_{\hat{\theta}}}{\partial u} \right)^T, \left(\frac{\partial L_{\hat{\theta}}}{\partial \lambda} \right)^T, \epsilon^T, \tilde{\theta}^T \right)^T$ and the estimated Lagrangian function:

$$L_{\hat{\theta}}(t, x, u, \lambda, \hat{\theta}) := J(t, x, u) + (k(t, x) - \Phi(t, x, u, \hat{\theta}))^T \lambda.$$

We will in what follows prove that the time and state-varying optimal set $\mathcal{O}_{u\lambda\tilde{\theta}}(t, x)$, in a certain sense is

UGS/UGAS with respect to (27) and the certainty equivalent control allocation algorithm:

$$\begin{pmatrix} \dot{u} \\ \dot{\lambda} \end{pmatrix} = -\Gamma \mathbb{H}_{\hat{\theta}} \begin{pmatrix} \frac{\partial L_{\hat{\theta}}}{\partial u} \\ \frac{\partial L_{\hat{\theta}}}{\partial \lambda} \end{pmatrix} - u_{ff\hat{\theta}} \quad (30)$$

$$\begin{aligned} \dot{\hat{\theta}} &= \Gamma_{\hat{\theta}}^{-1} (\Phi_\theta(t, x, u))^T (g_x(t, x))^T \left(\Gamma_\epsilon \epsilon + \left(\frac{\partial^2 L_{\hat{\theta}}}{\partial x \partial u} \right)^T \frac{\partial L_{\hat{\theta}}}{\partial u} \right) \\ &+ \Gamma_{\hat{\theta}}^{-1} (\Phi_\theta(t, x, u))^T (g_x(t, x))^T \left(\frac{\partial^2 L_{\hat{\theta}}}{\partial x \partial \lambda} \right)^T \frac{\partial L_{\hat{\theta}}}{\partial \lambda}. \end{aligned} \quad (31)$$

where $\mathbb{H}_{\hat{\theta}} := \begin{pmatrix} \frac{\partial^2 L_{\hat{\theta}}}{\partial u^2} & \frac{\partial^2 L_{\hat{\theta}}}{\partial \lambda \partial u} \\ \frac{\partial^2 L_{\hat{\theta}}}{\partial u \partial \lambda} & 0 \end{pmatrix}$, the matrices Γ , Γ_ϵ and $\Gamma_{\hat{\theta}}$, are symmetric positive definite and $u_{ff\hat{\theta}}$ is a feed-forward like term given by:

$$\begin{aligned} u_{ff\hat{\theta}} &:= \mathbb{H}_{\hat{\theta}}^{-1} \left(\begin{pmatrix} \frac{\partial^2 L}{\partial t \partial u} \\ \frac{\partial^2 L}{\partial t \partial \lambda} \end{pmatrix} + \begin{pmatrix} \frac{\partial^2 L}{\partial x \partial u} \\ \frac{\partial^2 L}{\partial x \partial \lambda} \end{pmatrix} f(t, x) + \begin{pmatrix} \frac{\partial^2 L_{\hat{\theta}}}{\partial \hat{\theta} \partial u} \\ \frac{\partial^2 L_{\hat{\theta}}}{\partial \hat{\theta} \partial \lambda} \end{pmatrix} \dot{\hat{\theta}} \right) \\ &+ \mathbb{H}_{\hat{\theta}}^{-1} \begin{pmatrix} \frac{\partial^2 L}{\partial x \partial u} \\ \frac{\partial^2 L}{\partial x \partial \lambda} \end{pmatrix} g(t, x) (k(t, x) - \Phi(t, x, u, \hat{\theta})) \end{aligned}$$

if $\det(\frac{\partial^2 L}{\partial u^2}) > \epsilon$ or $u_{ff\hat{\theta}} := 0$ if $\det(\frac{\partial^2 L}{\partial u^2}) < \epsilon$, where $(k_1)^r > \epsilon > 0$. k_1 is defined in the following assumption which both guarantee existence of the proposed update laws and the time-varying first order optimal set, $\mathcal{O}_{u\lambda}(t, x) := \left\{ (u^T, \lambda^T)^T \in \mathbb{R}^{r+d} \mid \left(\left(\frac{\partial L}{\partial u} \right)^T, \left(\frac{\partial L}{\partial \lambda} \right)^T \right)^T = 0 \right\}$.

Assumption 3 (Optimal Control Allocation)

- The cost function $J : \mathbb{R}_{\geq t_0} \times \mathbb{R}^{n \times r} \rightarrow \mathbb{R}$ is twice differentiable and $J(t, x, u) \rightarrow \infty$ as $|u| \rightarrow \infty$. Further there exist continuous increasing functions $\varsigma_{\partial J}, \varsigma_{\partial^2 J} : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ such that $\left| \frac{\partial J}{\partial u} \right| \leq \varsigma_{\partial J}(|(x^T, u^T)|)$ and $\left| \frac{\partial^2 J}{\partial t \partial u} \right| + \left| \frac{\partial^2 J}{\partial x \partial u} \right| \leq \varsigma_{\partial^2 J}(|(x^T, u^T)|) \forall t, x$ and u .
- There exists constants $k_2 > k_1 > 0$, such that $\forall t, x, \theta$ and $(u^T, \lambda^T)^T \notin \mathcal{O}_{u\lambda}^0$

$$k_1 I < \frac{\partial^2 L}{\partial u^2}(t, x, u, \lambda, \theta) \leq k_2 I, \quad (32)$$

where $\mathcal{O}_{u\lambda}^0$ is the interior of $\mathcal{O}_{u\lambda}$. If $(u^T, \lambda^T)^T \in \mathcal{O}_{u\lambda}^0$ then the lower bound is replaced by $\frac{\partial^2 L}{\partial u^2} \geq 0$.

Remark 4 The second order sufficient conditions in Theorem 12.6 in (Nocedal & Wright 1999) are satisfied for all t, x, u, λ and θ by Assumption 1 and 3, thus the set $\mathcal{O}_{u\lambda}(t, x)$ describes global optimal solutions of the problem (3).

Lemma 2 If Assumptions 1, 2 a) and 3 are satisfied, then $\mathcal{O}_{u\lambda}$ is non-empty for all t, x and θ . Further for all t, x, θ and $(u, \lambda) \in \mathcal{O}_{u\lambda}$, there exist a continuous function $\varsigma_{\mathcal{O}_{u\lambda}} : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ such that:

$$\left| (u^T, \lambda^T)^T \right| \leq \varsigma_{\mathcal{O}_{u\lambda}} \left(\left| (x^T, \theta^T)^T \right| \right)$$

PROOF. See Appendix A.

The idea of proving stability and convergence of the set $\mathcal{O}_{u\lambda\tilde{\theta}}(t, x)$, relies on the construction of the Lyapunov-like function:

$$V_{u\lambda\tilde{\theta}}(t, x, u, \lambda, \epsilon, \tilde{\theta}) := \frac{1}{2} \left(\left(\frac{\partial L_{\hat{\theta}}}{\partial u} \right)^T \frac{\partial L_{\hat{\theta}}}{\partial u} + \left(\frac{\partial L_{\hat{\theta}}}{\partial \lambda} \right)^T \frac{\partial L_{\hat{\theta}}}{\partial \lambda} \right) + \frac{\epsilon^T \Gamma_{\epsilon} \epsilon}{2} + \frac{\tilde{\theta}^T \Gamma_{\tilde{\theta}} \tilde{\theta}}{2}. \quad (33)$$

Its time derivative along the trajectories of the considered system is:

$$\begin{aligned} \dot{V}_{u\lambda\tilde{\theta}} = & \left(\left(\frac{\partial L_{\hat{\theta}}}{\partial u} \right)^T \frac{\partial^2 L_{\hat{\theta}}}{\partial u^2} + \left(\frac{\partial L_{\hat{\theta}}}{\partial \lambda} \right)^T \frac{\partial^2 L_{\hat{\theta}}}{\partial u \partial \lambda} \right) \dot{u} \\ & + \left(\frac{\partial L_{\hat{\theta}}}{\partial u} \right)^T \frac{\partial^2 L_{\hat{\theta}}}{\partial \lambda \partial u} \dot{\lambda} + \left(\left(\frac{\partial L_{\hat{\theta}}}{\partial u} \right)^T \frac{\partial^2 L_{\hat{\theta}}}{\partial x \partial u} + \left(\frac{\partial L_{\hat{\theta}}}{\partial \lambda} \right)^T \frac{\partial^2 L_{\hat{\theta}}}{\partial x \partial \lambda} \right) \dot{x} \\ & - \left(\left(\frac{\partial L_{\hat{\theta}}}{\partial u} \right)^T \frac{\partial^2 L_{\hat{\theta}}}{\partial \hat{\theta} \partial u} + \left(\frac{\partial L_{\hat{\theta}}}{\partial \lambda} \right)^T \frac{\partial^2 L_{\hat{\theta}}}{\partial \hat{\theta} \partial \lambda} \right) \dot{\hat{\theta}} \\ & + \left(\frac{\partial L_{\hat{\theta}}}{\partial u} \right)^T \frac{\partial^2 L_{\hat{\theta}}}{\partial t \partial u} + \left(\frac{\partial L_{\hat{\theta}}}{\partial \lambda} \right)^T \frac{\partial^2 L_{\hat{\theta}}}{\partial t \partial \lambda} + \epsilon^T \dot{\epsilon} + \tilde{\theta}^T \Gamma_{\tilde{\theta}} \dot{\tilde{\theta}}. \end{aligned}$$

Further by (27), (30) and (31), we get

$$\begin{aligned} \dot{V}_{u\lambda\tilde{\theta}} = & - \left(\left(\frac{\partial L_{\hat{\theta}}}{\partial u} \right)^T, \left(\frac{\partial L_{\hat{\theta}}}{\partial \lambda} \right)^T \right) \mathbb{H}_{\hat{\theta}} \Gamma_{\mathbb{H}_{\hat{\theta}}} \left(\left(\frac{\partial L_{\hat{\theta}}}{\partial u} \right)^T, \left(\frac{\partial L_{\hat{\theta}}}{\partial \lambda} \right)^T \right)^T \\ & - \epsilon^T \Gamma_{\epsilon} A \epsilon \\ \leq & -c_{\hat{\theta}} \left(\left(\frac{\partial L_{\hat{\theta}}}{\partial u} \right)^T \frac{\partial L_{\hat{\theta}}}{\partial u} + \left(\frac{\partial L_{\hat{\theta}}}{\partial \lambda} \right)^T \frac{\partial L_{\hat{\theta}}}{\partial \lambda} \right) - \epsilon^T \Gamma_{\epsilon} A \epsilon \quad (34) \end{aligned}$$

where $c_{\hat{\theta}} := \inf_t \lambda_{\min}(\mathbb{H}_{\hat{\theta}} \Gamma_{\mathbb{H}_{\hat{\theta}}}) > 0$. We formalize the result, based on set-stability.

Proposition 1 Let Assumptions 1, 2 a) and 3 be satisfied. Then if $x(t)$ exists for all t , the set $\mathcal{O}_{u\lambda\tilde{\theta}}(t, x(t))$ is UGS with respect to the system (27), (30) and (31), and $(\epsilon, \frac{\partial L_{\hat{\theta}}}{\partial u}, \frac{\partial L_{\hat{\theta}}}{\partial \lambda})$ converges asymptotically to zero.

PROOF. In order to prove this result we show that i) $\mathcal{O}_{u\lambda\tilde{\theta}}(t, x(t))$ is a closed forward invariant set with respect to system (27), (30) and (31), ii) the system (27), (30) and (31) is finite escape time detectable through $|z_{u\lambda\tilde{\theta}}|_{\mathcal{O}_{u\lambda\tilde{\theta}}}$, where $z_{u\lambda\tilde{\theta}} := (u^T, \lambda^T, \epsilon^T, \tilde{\theta}^T)^T$, and that iii) $\bar{V}_{u\lambda\tilde{\theta}}(t, z_{u\lambda\tilde{\theta}}) := V_{u\lambda\tilde{\theta}}(t, x(t), u, \lambda, \epsilon, \tilde{\theta})$, is a radially unbounded Lyapunov function.

i) Define $G(t, x, u, \lambda, \tilde{\theta}) := \left(\epsilon, \tilde{\theta}, \frac{\partial L_{\hat{\theta}}}{\partial u}^T, \frac{\partial L_{\hat{\theta}}}{\partial \lambda}^T \right)^T$. From Proposition 1.1.9 b) in (Bertsekas, Nedic &

Ozdaglar 2003) we have that $G : \mathbb{R}^q \rightarrow \mathbb{R}^{q_G}$ is continuous iff $G^{-1}(U)$ is closed in $\mathbb{R}^q$ for every closed U in $\mathbb{R}^{q_G}$. From the definition of $\mathcal{O}_{u\lambda\tilde{\theta}}$, $U = \{0\}$, and since G is continuous (by Assumption 1 - 3), $\mathcal{O}_{u\lambda\tilde{\theta}}$ is a closed set. The set is forward invariant if at t_1 , $G(t_1, x(t_1), u(t_1), \lambda(t_1), \tilde{\theta}(t_1)) = 0$ and $\frac{d(G(t, x, u, \lambda, \tilde{\theta}))}{dt} = 0 \forall t \geq t_1$ with respect to (27), (30), (31) and (28). Since there exist a continuous solution of (30) as long as x exist, we only need to check this condition on the boundary of $\mathcal{O}_{u\lambda\tilde{\theta}}(t, x(t))$ (Note that $\det(\mathbb{H}_{\hat{\theta}}) \neq 0$ on the boundary of $\mathcal{O}_{u\lambda\tilde{\theta}}(t, x(t))$ by Assumption 1d) and 3b)). We

get $G(t_1, x(t_1), u(t_1), \lambda(t_1), \tilde{\theta}(t_1)) = 0 \Rightarrow (\dot{\epsilon}, \dot{\tilde{\theta}}) = 0$ and it remains to prove that $\left(\frac{d\partial L_{\hat{\theta}}}{dt \partial \lambda}, \frac{d\partial L_{\hat{\theta}}}{dt \partial u} \right) = 0$.

$\frac{d\partial L_{\hat{\theta}}}{dt \partial \lambda} = \frac{\partial^2 L_{\hat{\theta}}}{\partial t \partial \lambda} + \frac{\partial^2 L_{\hat{\theta}}}{\partial x \partial \lambda} \dot{x} + \frac{\partial^2 L_{\hat{\theta}}}{\partial u \partial \lambda} \dot{u} + \frac{\partial^2 L_{\hat{\theta}}}{\partial \hat{\theta} \partial \lambda} \dot{\hat{\theta}}$ and $\frac{d\partial L_{\hat{\theta}}}{dt \partial u} = \frac{\partial^2 L_{\hat{\theta}}}{\partial t \partial u} + \frac{\partial^2 L_{\hat{\theta}}}{\partial x \partial u} \dot{x} + \frac{\partial^2 L_{\hat{\theta}}}{\partial u \partial u} \dot{u} + \frac{\partial^2 L_{\hat{\theta}}}{\partial \lambda \partial u} \dot{\lambda} + \frac{\partial^2 L_{\hat{\theta}}}{\partial \hat{\theta} \partial u} \dot{\hat{\theta}}$, thus

$$\left(\frac{d\partial L_{\hat{\theta}}}{dt \partial \lambda}, \frac{d\partial L_{\hat{\theta}}}{dt \partial u} \right) = \mathbb{H}_{\hat{\theta}} \begin{pmatrix} \dot{u} \\ \dot{\lambda} \end{pmatrix} + \mathbb{H}_{\hat{\theta}} u_{ff\hat{\theta}} = -\mathbb{H}_{\hat{\theta}} \Gamma_{\mathbb{H}_{\hat{\theta}}} \begin{pmatrix} \frac{\partial L_{\hat{\theta}}}{\partial u} \\ \frac{\partial L_{\hat{\theta}}}{\partial \lambda} \end{pmatrix} = 0.$$

ii) Since $x(t)$ is assumed to be forward complete, it follows from Lemma 2 that there always exists a pair $(u^{*T}, \lambda^{*T})^T$ bounded by x and $\hat{\theta} = \theta - \tilde{\theta}$ that satisfy the time-varying first order optimal conditions $\left(\frac{\partial L_{\hat{\theta}}}{\partial u}^T, \frac{\partial L_{\hat{\theta}}}{\partial \lambda}^T \right) = 0$, and thus the system (27), (30) and (31) is finite escape time detectable through $|z_{u\lambda\tilde{\theta}}|_{\mathcal{O}_{u\lambda\tilde{\theta}}}$.

iii) In the proof of Lemma, change L with $L_{\hat{\theta}}$ such that (A.3) become

$$\left(\frac{\partial L_{\hat{\theta}}}{\partial u} \right)^T \frac{\partial L_{\hat{\theta}}}{\partial u} + \left(\frac{\partial L_{\hat{\theta}}}{\partial \lambda} \right)^T \frac{\partial L_{\hat{\theta}}}{\partial \lambda} = \begin{pmatrix} u - u^* \\ \lambda - \lambda^* \end{pmatrix}^T \mathbb{H}_{\hat{\theta}^*}^T \mathbb{H}_{\hat{\theta}^*} \begin{pmatrix} u - u^* \\ \lambda - \lambda^* \end{pmatrix} \quad (35)$$

and it follows that there exists constants $\kappa_2 \geq \kappa_1 > 0$ such that $\bar{\kappa}_1 |z_{u\lambda}|_{\mathcal{O}_{u\lambda}}^2 \leq \left(\left(\frac{\partial L_{\hat{\theta}}}{\partial u} \right)^T \frac{\partial L_{\hat{\theta}}}{\partial u} + \left(\frac{\partial L_{\hat{\theta}}}{\partial \lambda} \right)^T \frac{\partial L_{\hat{\theta}}}{\partial \lambda} \right) \leq \bar{\kappa}_2 |z_{u\lambda}|_{\mathcal{O}_{u\lambda}}^2$ since $\mathbb{H}_{\hat{\theta}^*}^T \mathbb{H}_{\hat{\theta}^*}$ is positive definite and bounded from Assumption 1 and 3. By a similar argument, in (A.4) change P_{*c} with $\frac{\partial^2 L_{\hat{\theta}}}{\partial u^2}$, $\mathbb{H}_{\hat{\theta}}$ is non-singular, hence the control allocation law (30) always exist and, $\kappa_1 |z_{u\lambda\tilde{\theta}}|_{\mathcal{O}_{u\lambda\tilde{\theta}}} \leq \bar{V}_{u\lambda\tilde{\theta}}(t, z_{u\lambda\tilde{\theta}}) \leq \kappa_2 |z_{u\lambda\tilde{\theta}}|_{\mathcal{O}_{u\lambda\tilde{\theta}}}$, where $\kappa_1 := \frac{1}{2} \min(\bar{\kappa}_1, \Gamma_{\epsilon}, \Gamma_{\tilde{\theta}})$ $\kappa_2 := \frac{1}{2} \max(\bar{\kappa}_2, \Gamma_{\epsilon}, \Gamma_{\tilde{\theta}})$, such that $\bar{V}_{u\lambda\tilde{\theta}}(t, z_{u\lambda\tilde{\theta}})$ is a radially unbounded Lyapunov function.

Since (34) is negative semidefinite, UGS of the set $\mathcal{O}_{u\lambda\tilde{\theta}}$ can be concluded by Theorem 1. The convergence result follows by Barbalat's lemma ■

Corollary 2 Let Assumptions 1, 2 a) and 3 be satisfied and $|z_{u\lambda\tilde{\theta}0}|_{\mathcal{O}_{u\lambda\tilde{\theta}}} < r$ where $r > 0$. If $x(t)$ exists

for $t \in [t_0, T]$ where $T \geq t_0$, then there exists a positive constant $B(r) > 0$ such that for all $t \in [t_0, T]$, $|z_{u\lambda\bar{\theta}}(t)|_{\mathcal{O}_{u\lambda\bar{\theta}}(t, x(t))} \leq B(r)$.

PROOF. Since $\dot{\bar{V}}_{u\lambda\bar{\theta}} \leq 0$ and $\bar{V}_{u\lambda\bar{\theta}}(t, z_{u\lambda\bar{\theta}}(t)) \leq \bar{V}_{u\lambda\bar{\theta}}(t_0, z_{u\lambda\bar{\theta}}(t_0))$ for all $t \in [t_0, T]$. From iii) in the proof of Proposition 1 $\kappa_1 |z_{u\lambda\bar{\theta}}|_{\mathcal{O}_{u\lambda\bar{\theta}}} \leq \bar{V}_{u\lambda\bar{\theta}}(t, z_{u\lambda\bar{\theta}}) \leq \kappa_2 |z_{u\lambda\bar{\theta}}|_{\mathcal{O}_{u\lambda\bar{\theta}}}$ for all $t \in [t_0, T]$, thus it follows that $|z_{u\lambda\bar{\theta}}(t)|_{\mathcal{O}_{u\lambda\bar{\theta}}(t, x(t))} \leq \sqrt{\frac{1}{\kappa_1} \bar{V}_{u\lambda\bar{\theta}}(t, z_{u\lambda\bar{\theta}}(t))} \leq \sqrt{\frac{1}{\kappa_1} \bar{V}_{u\lambda\bar{\theta}}(t_0, z_{u\lambda\bar{\theta}}(t_0))} \leq \sqrt{\frac{\kappa_2}{\kappa_1}} |z_{u\lambda\bar{\theta}}|_{\mathcal{O}_{u\lambda\bar{\theta}}} =: B(r)$ ■

Remark 5 Provided that the gain matrix, $\Gamma > 0$, is bounded away from zero, Γ may be chosen time-varying. If for example $\Gamma = \gamma (\mathbb{H}_{\bar{\theta}} \mathbb{H}_{\bar{\theta}})^{-1}$ for some $\gamma > 0$, then

$$\begin{pmatrix} \dot{u} \\ \dot{\lambda} \end{pmatrix} = -\gamma \mathbb{H}_{\bar{\theta}}^{-1} \begin{pmatrix} \frac{\partial L_{\bar{\theta}}}{\partial u} \\ \frac{\partial L_{\bar{\theta}}}{\partial \lambda} \end{pmatrix} - \mathbb{H}_{\bar{\theta}}^{-1} u_{ff}$$

where the first term is the Newton direction when $L_{\bar{\theta}}$ is considered the cost function to be minimized. In case $\mathbb{H}_{\bar{\theta}} \mathbb{H}_{\bar{\theta}}$ is poorly conditioned one may choose $\Gamma = \gamma (\mathbb{H}_{\bar{\theta}} \mathbb{H}_{\bar{\theta}} + \rho I)^{-1}$ for some $\rho > 0$, to avoid $c_{\bar{\theta}}$ in (34) from being small.

Assumption 4 (Persistence of Excitation) The signal matrix $\Phi_g(t) := g_x(t, x(t)) \Phi_{\theta}(t, x(t), u(t))$ is Persistently Excited (PE), which means that there exist constants T and $\gamma > 0$, such that

$$\int_t^{t+T} \Phi_g(\tau)^T \Phi_g(\tau) d\tau \geq \gamma I, \forall t > t_0.$$

Remark 6 The system trajectories $x(t)$ defined by (1) typically represents the tracking error i.e. $x := x_s - x_d$, where x_s is states vector of the system, and x_d represents the desired reference for these states. This means that PE assumption on $\Phi_g(t)$ is dependent on the reference trajectory x_d , in addition to disturbances and noise, and imply that some "richness" properties of these signals are satisfied.

Proposition 2 Let $x(t)$ be UGB, then if Assumption 4 and the assumptions of Proposition 1 are satisfied, the set $\mathcal{O}_{u\lambda\bar{\theta}}$ is UGAS with respect to system (26), (27), (30) and (31).

PROOF. See Appendix B ■

Unless the PE condition is satisfied for Φ_g , only stability of the optimal set is guaranteed. Thus in the sense of achieving asymptotic optimality, parameter convergence is of importance.

3.2.2 Stability of the combined control and certainty equivalent control allocation

The optimal set for the combined control and certainty equivalent control allocation problem is define by:

$$\mathcal{O}_{xu\lambda\bar{\theta}}(t) := \mathcal{O}_x(t) \times \mathcal{O}_{u\lambda\bar{\theta}}(t, 0). \quad (36)$$

In the framework of cascaded systems, we consider (30), (31) together with (27) to be the perturbing system (Σ_2), while (28) represents the perturbed system (Σ_1). Loosely explained, we will use Lemma 1 to conclude UGAS of the set $\mathcal{O}_{xu\lambda\bar{\theta}}$ if, $\mathcal{O}_x$ and $\mathcal{O}_{u\lambda\bar{\theta}}$ individually are UGAS (which we already have established) and the combined system is UGB with respect to $\mathcal{O}_{xu\lambda\bar{\theta}}$.

Before establishing the main results of this section, we need to state an assumption on the interconnection term between the two systems. We start by stating the following property:

Lemma 3 By Assumption 1 and Lemma 2 there exists continuous functions $\varsigma_x, \varsigma_{xu}, \varsigma_u : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$, such that

$$|\Phi_{\theta}(t, x, u)| \leq \varsigma_x(|x|) \varsigma_{xu}(|x|) + \varsigma_x(|x|) \varsigma_u(|z_{u\lambda\bar{\theta}}|_{\mathcal{O}_{u\lambda\bar{\theta}}}).$$

PROOF. This can be seen by applying (Mazenc & Praly 1996)'s lemma B.1 ■

Assumption 2 (Continued)

c) The function, α_k , from Assumption 2 b) has the following additional property:

$$\alpha_k^{-1}(|x|) \alpha_{x3}(|x|) \geq \alpha_{x4}(|x|) \varsigma_{x \max}(|x|), \quad (37)$$

$$\text{where } \varsigma_{x \max}(|x|) := \max(1, \varsigma_x(|x|), \varsigma_x(|x|) \varsigma_{xu}(|x|)).$$

Remark 7 If there exists a constant $K > 0$ such that $\varsigma_{x \max}(|x|) \leq K \forall x$, which is common in a mechanical system, Assumption 2 c) is satisfied.

Next we consider the closed loop of the plant, the virtual controller and the adaptive dynamic control allocation law:

Proposition 3 If Assumptions 1-3 are satisfied, then the set $\mathcal{O}_{xu\lambda\bar{\theta}}$ is UGS, with respect to system (28), (30), (31) and (27). If in addition Assumption 4 is satisfied, then $\mathcal{O}_{xu\lambda\bar{\theta}}$ is UGAS with respect to (28), (30), (31) and (27).

PROOF. The main part of this proof is to prove boundedness, completeness and invoke Lemma 1. Let $|z_{xu\lambda\bar{\theta}}|_{\mathcal{O}_{xu\lambda\bar{\theta}}} \leq r$, where $r > 0$, and assume that $|z_x(t)|_{\mathcal{O}_x}$ escapes to infinity at T . Then for any constant $M(r)$ there exists a $t \in [t_0, T]$ such that $M(r) \leq |z_x(t)|_{\mathcal{O}_x}$. In what follows we show that $M(r)$ can not be chosen arbitrarily. Define $v(t, z_x) := V_x(t, x)$ such that

$$\begin{aligned}
\dot{v} &\leq -\alpha_{x3}(|z_x|_{\mathcal{O}_x}) + \frac{\partial V_x}{\partial x} g(t, x) \left(k(t, x) - \Phi(t, x, u, \hat{\theta}) \right) \\
&+ \frac{\partial V_x}{\partial x} g_x(t, x) \Phi_\theta(t, x, u) \tilde{\theta} \\
&\leq -\alpha_{x3}(|z_x|_{\mathcal{O}_x}) + \left| \frac{\partial V_x}{\partial x} \right| |g(t, x)| \left| \frac{\partial L_{\hat{\theta}}}{\partial \lambda}(t, x, u, \hat{\theta}) \right| \\
&+ \left| \frac{\partial V_x}{\partial x} \right| |g_x(t, x)| \left| \Phi_\theta(t, x, u) \tilde{\theta} \right| \\
&\leq -\alpha_{x3}(|z_x|_{\mathcal{O}_x}) \\
&+ \alpha_{x4}(|z_x|_{\mathcal{O}_x}) K(\kappa_2 + |\Phi_\theta(t, x, u)|) |z_{u\lambda\tilde{\theta}}|_{\mathcal{O}_{u\lambda\tilde{\theta}}} . \quad (38)
\end{aligned}$$

From Corollary 2, there exists a positive constant $B(r) \geq 0$, for all $t \in [t_0, T)$, such that for $|z_{u\lambda\tilde{\theta}0}|_{\mathcal{O}_{u\lambda\tilde{\theta}}} \leq r$, $|z_{u\lambda\tilde{\theta}}(t)|_{\mathcal{O}_{u\lambda\tilde{\theta}}} \leq B(|z_{u\lambda\tilde{\theta}0}|_{\mathcal{O}_{u\lambda\tilde{\theta}}}) \leq B(r)$. From Assumption 1 and 2,

$$\begin{aligned}
\dot{v} &\leq -\alpha_{x3}(|z_x|_{\mathcal{O}_x}) \\
&+ \alpha_{x4}(|z_x|_{\mathcal{O}_x}) K(\kappa_2 + |\Phi_\theta(t, x, u)|) B(r) \\
&\leq -\alpha_k(|z_x|_{\mathcal{O}_x}) \alpha_{x4}(|z_x|_{\mathcal{O}_x}) \varsigma_x \max(|z_x|_{\mathcal{O}_x}) \\
&+ \alpha_{x4}(|z_x|_{\mathcal{O}_x}) K(\kappa_2 + |\Phi_\theta(t, x, u)|) B(r) \\
&\leq -\alpha_{x4}(|z_x|_{\mathcal{O}_x}) \alpha_k(|z_x|_{\mathcal{O}_x}) \varsigma_x \max(|z_x|_{\mathcal{O}_x}) \\
&+ \alpha_{x4}(|z_x|_{\mathcal{O}_x}) K\kappa_2 B(r) \\
&+ \alpha_{x4}(|z_x|_{\mathcal{O}_x}) K\varsigma_x(|z_x|_{\mathcal{O}_x}) \varsigma_{xu}(|z_x|_{\mathcal{O}_x}) B(r) \\
&+ \alpha_{x4}(|z_x|_{\mathcal{O}_x}) K\varsigma_x(|z_x|_{\mathcal{O}_x}) \varsigma_u(B(r)) B(r) \quad (39)
\end{aligned}$$

$$\begin{aligned}
&\leq -\alpha_{x4}(|z_x|_{\mathcal{O}_x}) \alpha_k(\alpha_{x2}^{-1}(v)) \varsigma_x \max(|z_x|_{\mathcal{O}_x}) \\
&+ \alpha_{x4}(|z_x|_{\mathcal{O}_x}) K\kappa_2 B(r) \\
&+ \alpha_{x4}(|z_x|_{\mathcal{O}_x}) K\varsigma_x(|z_x|_{\mathcal{O}_x}) \varsigma_{xu}(|z_x|_{\mathcal{O}_x}) B(r) \\
&+ \alpha_{x4}(|z_x|_{\mathcal{O}_x}) K\varsigma_x(|z_x|_{\mathcal{O}_x}) \varsigma_u(B(r)) B(r). \quad (40)
\end{aligned}$$

Thus, if

$$|z_{x0}|_{\mathcal{O}_x} > \alpha_k^{-1}(K(\kappa_2 + 1 + \varsigma_u(B(r))) B(r))$$

then from (39),

$$v(t_0, z_{x0}) \geq v(t, z_x(t)) \text{ and } |z_x(t)|_{\mathcal{O}_x} \leq \alpha_{x1}^{-1}(v(z_{x0})) \leq \alpha_{x1}^{-1}(\alpha_{x2}(r)), \text{ else,}$$

$$|z_{x0}|_{\mathcal{O}_x} \leq \alpha_k^{-1}(K(\kappa_2 + 1 + \varsigma_u(B(r))) B(r))$$

and from (40),

$$v(t, z_x(t)) \leq \alpha_{x2}(\alpha_k^{-1}(K(\kappa_2 + 1 + \varsigma_u(B(r))) B(r)))$$

such that

$$|z_x(t)|_{\mathcal{O}_x} \leq \alpha_{x1}^{-1}(\alpha_{x2}(\alpha_k^{-1}(K(\kappa_2 + 1 + \varsigma_u(B(r))) B(r)))) .$$

By choosing

$$\begin{aligned}
M(r) &:= \max(\alpha_{x1}^{-1}(\alpha_{x2}(r)), \\
&\alpha_{x1}^{-1}(\alpha_{x2}(\alpha_k^{-1}(K(\kappa_2 + 1 + \varsigma_u(B(r))) B(r))))),
\end{aligned}$$

the assumption of $|z_x(t)|_{\mathcal{O}_x}$ escaping to infinity is contradicted, since $M(r) > |z_x(t)|_{\mathcal{O}_x}$ and $|\cdot|_{\mathcal{O}_x}$ is finite escape time detectable. Furthermore $\mathcal{O}_x$ is UGB. From Propositions 1 and 2 and the assumptions of these propositions, the assumptions of Lemma 1 and Corollary 1 are satisfied and the result is proved ■

Proposition 3 implies that the time-varying optimal set $\mathcal{O}_{xu\lambda\tilde{\theta}}(t)$ is uniformly stable, and in addition uniformly attractive if Assumption 4 is satisfied. Thus optimal control allocation is achieved asymptotically for the closed loop.

A local version of this result can be proven using Corollary 1.

Corollary 3 *If for $u \in \mathbb{U} \subset \mathbb{R}^r$ there exist constant $c_x > 0$ such that for $|x| \leq c_x$ the domain $\mathbb{U}_z \subset \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{U} \times \mathbb{R}^{d+n+m}$ contain $\mathcal{O}_{xu\lambda\tilde{\theta}}$, then if the Assumptions 1-3 are satisfied, the set $\mathcal{O}_{xu\lambda\tilde{\theta}}$ is US with respect to the system (28), (30), (31) and (27). If in addition Assumption 4 is satisfied, $\mathcal{O}_{xu\lambda\tilde{\theta}}$ is UAS with respect to the system (28), (30), (31) and (27).*

PROOF. Since $\mathcal{O}_{xu\lambda\tilde{\theta}} \subset \mathbb{U}_z$, there exist a positive constants $r \geq |z_{xu\lambda\tilde{\theta}0}|_{\mathcal{O}_{xu\lambda\tilde{\theta}}}$ such that $|z_x(t)|_{\mathcal{O}_x} \leq c_{x0}$ where $0 < c_{x0} < c_x$, hence the domain $\mathbb{U}_{u\lambda\tilde{\theta}} \subset \mathbb{R}_{\geq 0} \times \mathbb{U} \times \mathbb{R}^{d+n+m}$ contain $\mathcal{O}_{u\lambda\tilde{\theta}}$. US/UAS of $\mathcal{O}_{u\lambda\tilde{\theta}}$ follows from the Lyapunov-like function $V_{u\lambda\tilde{\theta}}$ and the PE Assumption. UB follows from $|z_{xu\lambda\tilde{\theta}0}|_{\mathcal{O}_{xu\lambda\tilde{\theta}}} \leq r$, and the US/UAS property of $\mathcal{O}_{xu\lambda\tilde{\theta}}$ follows from Corollary 1 ■

Corollary 4 *If θ is known and the Assumptions 1-3 are satisfied then the set $\mathcal{O}_{xu\lambda} := \mathcal{O}_x(t) \times \mathcal{O}_{u\lambda\tilde{\theta}}(t, 0)$ is UGAS with respect to the system (19) and (30).*

PROOF. This result follows by working through the above arguments, neglecting the adaptive dynamic and considering a Lyapunov like function of the form $V_{u\lambda}(t, x, u, \lambda) := \frac{1}{2} \left(\left(\frac{\partial L_{\hat{\theta}}}{\partial u} \right)^T \frac{\partial L_{\hat{\theta}}}{\partial u} + \left(\frac{\partial L_{\hat{\theta}}}{\partial \lambda} \right)^T \frac{\partial L_{\hat{\theta}}}{\partial \lambda} \right)$ ■

Corollary 5 *If for $u \in \mathbb{U} \subset \mathbb{R}^r$ there exist constant $c_x > 0$ such that for $|x| \leq c_x$ the domain $\mathbb{U}_z \subset \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{U} \times \mathbb{R}^d$ contain $\mathcal{O}_{xu\lambda}$, then if the Assumptions 1-3 are satisfied, the set $\mathcal{O}_{xu\lambda}$ is UAS with respect to the system (19) and (30).*

Since the set with respect to system Σ_2 may only be US, due to actuator/effector constraints and parameter uncertainty, only US of the cascade, may be concluded. But if the PE property is satisfied on Φ_g , a UAS result may be achieved with both optimal and adaptive convergence.

4 Example

In this section simulation results of an over-actuated scaled-model ship, manoeuvred at low-speed, is presented. A 3DOF horizontal plane model described by:

$$\begin{aligned}
\dot{\eta}_e &= R(\psi)\nu \\
\dot{\nu} &= -M^{-1}D\nu + M^{-1}(\tau + b) \\
\tau &= \Phi_\theta(u)\theta,
\end{aligned} \quad (41)$$

is considered, where $\eta_e := (x_e, y_e, \psi_e)^T := (x_p - x_d, y_p - y_d, \psi_p - \psi_d)^T$ is the north and east positions and compass heading deviation. Subscript p and d denotes the

actual and desired states. $\nu := (u, v, r)^T$ is the body-fixed velocities, surge, sway and yaw, τ is the generalized force vector, $b := (b_1, b_2, b_3)^T$ is a disturbance due to wind and current and $R(\psi)$ is the rotation matrix function between the body fixed and the earth fixed coordinate frame. The example we present here is based on (Lindegaard & Fossen 2003), and is also studied in (Johansen 2004) and (Tjønnås & Johansen 2005). In the considered model there are five force producing devices; the two main propellers aft of the hull, in conjunction with two rudders, and one tunnel thruster going through the hull of the vessel. ω_i denotes the propeller angular velocity and δ_i denotes the rudder deflection. $i = 1, 2$ denotes the aft actuators, while $i = 3$ denotes the tunnel thruster. (41) can be rewritten in the form of (1) and (2) by:

$$\begin{aligned} x &:= (\xi, \eta_e, \nu)^T, \quad \theta := (\theta_1, \theta_2)^T, \\ \tau &:= (\tau_1, \tau_2, \tau_3)^T, \quad u := (\omega_1, \omega_2, \omega_3, \delta_1, \delta_2)^T, \\ \Phi_\theta(u) &:= \begin{pmatrix} X_1 + X_2 & 0 \\ Y_1 + Y_2 & T_3 \\ \Phi_{\theta 13} & l_{3,x}T_3 \end{pmatrix}, \\ \Phi_{\theta 13} &:= -l_{1,y}X_1 + l_{1,x}Y_1 - l_{2,y}X_2 + l_{2,x}Y_2, \\ X_i &:= T_i - D_i, \quad Y_i := L_i, \end{aligned}$$

where the thruster forces are defined by:

$$\begin{aligned} T_i &:= \begin{cases} k_{Tp_i}\omega_i^2 & \omega_i \geq 0 \\ k_{Tn_i}|\omega_i|\omega_i & \omega_i < 0 \end{cases}, \\ L_i &:= \begin{cases} T_i(1 + k_{Ln_i}\omega_i)(k_{L\delta 1_i} + k_{L\delta 2_i}|\delta_i|)\delta_i, & \omega_i \geq 0 \\ 0 & \omega_i < 0 \end{cases}, \\ D_i &:= \begin{cases} T_i(1 + k_{Dn_i}\omega_i)(k_{D\delta 1_i}|\delta_i| + k_{D\delta 2_i}\delta_i^2), & \omega_i \geq 0 \\ 0 & \omega_i < 0 \end{cases}. \end{aligned}$$

The unknown parameter vector θ represents thrust losses. θ_2 is also related to the parameters k_{Tp_3} and k_{Tn_3} in a multiplicative way. This suggest that the estimate of θ_2 gives a direct estimate of the tunnel thruster loss factor. The virtual controller

$$\tau_c := -K_i R^T(\psi)\xi - K_p R^T(\psi)\eta_e - K_d \nu,$$

proposed in (Lindegaard & Fossen 2003), stabilizes the system (41) augmented with: $\dot{\zeta} = -\eta_e$, where $\zeta := b - \xi$ and $\dot{\xi} = \eta_e$, uniformly and exponentially, for some physically limited yaw rate. The cost function used is:

$$\begin{aligned} J(u) &:= \sum_{i=1}^3 k_i |\omega_i| \omega_i^2 + k_{i2} \omega_i^2 + \sum_{i=1}^2 q_i \delta_i^2, \\ |\omega_i| &\leq 18Hz, \quad |\delta_i| \leq 35 \text{ deg}, \\ k_1 &= k_2 = 0.01, \quad k_3 = 0.02, \quad q_1 = q_2 = 500, \\ k_{i2} &= 10^{-3}. \end{aligned}$$

The gain matrices are chosen as follows: $K_p := M \cdot \text{diag}(3.13, 3.13, 12.5)10^{-2}$, $K_d := M \cdot \text{diag}(3.75, 3.75, 7.5)10^{-1}$, $K_I := M \cdot \text{diag}(0.2, 0.2, 4)10^{-3}$, $A_\epsilon := I_{9 \times 9}$, $Q_\theta := \text{diag}(1, 1)$, $Q_\epsilon := \text{diag}(a, 150 \cdot (100, 100, 1)^T)$, $a := (1, 1, 1, 1, 1, 1)^T$ and $\Gamma := \left(\mathbb{H}_\theta^T W \mathbb{H}_\theta + \varepsilon I \right)^{-1}$ where $W := \text{diag}(1, 1, 1, 1, 1, 0.6, 0.6, 0.6)$ and $\varepsilon := 10^{-9}$. The weighting matrix W is chosen such that the deviation of $\left| \frac{\partial L_\theta}{\partial \lambda} \right| = \left| k(t, x) - \Phi(t, x, u, \hat{\theta}) \right|$ from zero, is penalized more then the deviation of $\left| \frac{\partial L_\theta}{\partial u} \right|$ from zero, in the search direction. In order to keep $\hat{\theta}$ from being zero, due to a physical consideration, a projection algorithm can be used. i.e. from (30)

$$\dot{\hat{\theta}} = \begin{cases} \Gamma_\theta^{-1} \xi & \text{(i)} \\ \Gamma_\theta^{-1} \xi + \Gamma_\theta^{-1} \frac{(1,1)^T (1,1)}{(1,1) \Gamma_\theta^{-1} (1,1)^T} \Gamma_\theta^{-1} \xi & \text{(ii)} \end{cases}$$

where (i) is used if $\hat{\theta} \in S^0$ or $\theta \in \delta(S)$ and $\left(\Gamma_\theta^{-1} \xi \right)^T (1, 1)^T \leq 0$, and (ii) is used otherwise. Further

$$\xi := \Phi_\theta(u)^T g_x^T \left(\Gamma_\epsilon \epsilon + \frac{\partial^2 L_\theta}{\partial x \partial u} \frac{\partial L_\theta}{\partial u} + \frac{\partial^2 L_\theta}{\partial x \partial \lambda} \frac{\partial L_\theta}{\partial \lambda} \right),$$

$S := \left\{ \hat{\theta} \in \mathbb{R}^2 \mid -\min_i \{\hat{\theta}_i\} \leq \hat{\theta}_{\min} \right\}$, S^0 is the interior of S and $\delta(S)$ is the boundary of S and $\hat{\theta}_{\min} > 0$.

Remark 8 In order to satisfy Assumption 1 we replace τ in (41) by $\tau = \Phi(u, \theta) = \Phi_\theta(u)\theta + \Phi_0(u)$, where $\Phi_0(u) := \varsigma (\omega_1, \omega_2, \omega_3)^T$ and $\varsigma > 0$, in the control allocation algorithm. Since ς may be chosen arbitrarily small, this actuator/effector mapping is practically similar to (41), but ensures that the allocation algorithm is well conditioned. Assumption 3 is satisfied locally since for bounded u and λ , k_{i2} ensures that $\frac{\partial^2 L}{\partial u^2} > 0$. It is also worth noticing that the optimal control allocation solution in this example, at any time, is a single point.

Based on the wind and current disturbance vector $b := 0.05(1, 1, 1)^T$ and the parameter vector $\theta := (0.8, 0.9)^T$, the simulation results are presented in the Figures 3-7. Due to the disturbance and reference change, the transients excite the parameter update-law at $t \approx 0$ as well as at $t \approx 200$ and $t \approx 400$, and the estimated parameters converges to the true values. The control objective is satisfied and the commanded virtual controls are tracked closely, by the forces generated by the adaptive control allocation law, except for $t \approx 0$, $t \approx 200$ and $t \approx 400$, where the control allocation is suboptimal due to initial conditions and actuator saturation: see Figure 7. The simulations are carried out in the MATLAB environment with a sampling rate of 10Hz.

Fig. 3. Simulation results - the solid lines represent positions while the dashed lines represent references

Fig. 4. Simulation results - computed propeller angular velocities by the allocation algorithm

5 Conclusion

Based on a control-Lyapunov design approach, an adaptive optimizing nonlinear control allocation algorithm is derived. Under certain assumptions on the system (actuator/effector model) and the control design (growth rate conditions on the Lyapunov function), a cascade result is used to prove, closed-loop, stability and attractivity of a set representing the optimal actuator configuration. Typical applications for the control allocation algorithm are over-actuated mechanical systems, especially systems that exhibits fast dynamics since the algorithm is computational efficient. An automotive example is presented in (Tjønnås & Johansen 2006).

Fig. 5. Simulation results - computed rudder deflection by the allocation algorithm

Fig. 6. Simulation results - the parameter adaption

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Fig. 7. Simulation results - the solid lines represent actual forces generated by the actuators, the dashed lines represent the virtual forces generated by the control law.

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A Proof of Lemma 2

The boundary of the set $\mathcal{O}_{u\lambda}$ and the time-varying first order optimal solution is described by:

$$\frac{\partial L}{\partial u}(t, x, u, \lambda, \theta) = \frac{\partial J}{\partial u}(t, x, u) - \left(\frac{\partial \Phi}{\partial u}(t, x, u, \theta) \right)^T \lambda$$

$$\frac{\partial L}{\partial \lambda}(t, x, u, \lambda, \theta) = k(t, x) - \Phi(t, x, u, \theta)$$

By using theorem 2.4.7 in (Abrahamson, Marsden & Ratiu 1988), it can be shown that

$$\begin{aligned} \frac{\partial L}{\partial u}(t, x, u, \lambda, \theta) - \frac{\partial L}{\partial u}(t, x, u_c, \lambda, \theta) &= \frac{\partial^2 L}{\partial u^2} \Big|_c (u - u_c) \\ \frac{\partial L}{\partial u}(t, x, u_c, \lambda, \theta) - \frac{\partial L}{\partial u}(t, x, u_c, \lambda^*, \theta) &= \frac{\partial^2 L}{\partial \lambda \partial u} \Big|_c (\lambda - \lambda^*) \\ \frac{\partial L}{\partial u}(t, x, u_c, \lambda^*, \theta) - \frac{\partial L}{\partial u}(t, x, u^*, \lambda^*, \theta) &= \frac{\partial^2 L}{\partial u^2} \Big|_* (u_c - u^*) \\ \frac{\partial L}{\partial \lambda}(t, x, u, \lambda, \theta) - \frac{\partial L}{\partial \lambda}(t, x, u^*, \lambda, \theta) &= \frac{\partial^2 L}{\partial u \partial \lambda} \Big|_* (u - u^*) \\ \frac{\partial L}{\partial \lambda}(t, x, u^*, \lambda, \theta) - \frac{\partial L}{\partial \lambda}(t, x, u^*, \lambda^*, \theta) &= 0. \end{aligned}$$

where $\frac{\partial^2 L}{\partial u^2} \Big|_c := \int_0^1 \frac{\partial^2 L}{\partial u^2}(t, x, (1-s)u_c + su, \lambda, \theta) ds$,
 $\frac{\partial^2 L}{\partial u^2} \Big|_* := \int_0^1 \frac{\partial^2 L}{\partial u^2}(t, x, (1-s)u^* + su_c, \lambda, \theta) ds$, $\frac{\partial^2 L}{\partial \lambda \partial u} \Big|_c := \int_0^1 \frac{\partial^2 L}{\partial \lambda \partial u}(t, x, u_c, (1-s)\lambda^* + s\lambda, \theta) ds = -\frac{\partial \Phi}{\partial u}^T(t, x, u_c, \theta)$
and $\frac{\partial^2 L}{\partial u \partial \lambda} \Big|_* := \int_0^1 \frac{\partial^2 L}{\partial u \partial \lambda}(t, x, (1-s)u^* + su, \lambda, \theta) ds = -\frac{\partial \Phi}{\partial u}(t, x, u_c, \theta)$. Since
 $\frac{\partial L}{\partial \lambda}(t, x, u^*, \lambda^*, \theta) = 0$ and $\frac{\partial L}{\partial u}(t, x, u^*, \lambda^*, \theta) = 0$, we get

$$\frac{\partial L}{\partial u}(t, x, u, \lambda, \theta) = \frac{\partial^2 L}{\partial u^2} \Big|_c (u - u_c) + \frac{\partial^2 L}{\partial u^2} \Big|_* (u_c - u^*) - \frac{\partial \Phi}{\partial u}^T(t, x, u_c, \theta)(\lambda - \lambda^*) \quad (\text{A.1})$$

$$\frac{\partial L}{\partial \lambda}(t, x, u, \lambda, \theta) = -\frac{\partial \Phi}{\partial u}(t, x, u_c, \theta)(u - u^*), \quad (\text{A.2})$$

and from $u_c := \vartheta(u - u^*) + u^*$ where $\vartheta := \text{diag}(\vartheta_i)$ and $0 < \vartheta_i < 1$:

$$\begin{aligned} & \frac{\partial^2 L}{\partial u^2} \Big|_c (u - u_c) + \frac{\partial^2 L}{\partial u^2} \Big|_* (u_c - u^*) \\ &= (1 - \vartheta) \frac{\partial^2 L}{\partial u^2} \Big|_c (u - u^*) + \vartheta \frac{\partial^2 L}{\partial u^2} \Big|_* (u - u^*). \end{aligned}$$

From Assumption 3, $\frac{\partial^2 L}{\partial u^2} \Big|_c$ and $\frac{\partial^2 L}{\partial u^2} \Big|_*$ are positive definite matrices, such that $P_{*c} := (1 - \vartheta) \frac{\partial^2 L}{\partial u^2} \Big|_c + \vartheta \frac{\partial^2 L}{\partial u^2} \Big|_*$ is also a positive definite matrix. Further

$$\left(\frac{\partial L}{\partial u} \right)^T \frac{\partial L}{\partial u} + \left(\frac{\partial L}{\partial \lambda} \right)^T \frac{\partial L}{\partial \lambda} = \begin{pmatrix} u - u^* \\ \lambda - \lambda^* \end{pmatrix}^T \mathbb{H}_*^T \mathbb{H}_* \begin{pmatrix} u - u^* \\ \lambda - \lambda^* \end{pmatrix} \quad (\text{A.3})$$

where $\mathbb{H}_* = \begin{pmatrix} P_{*c} & \frac{\partial \Phi}{\partial u}^T(t, x, u_c, \theta) \\ \frac{\partial \Phi}{\partial u}(t, x, u_c, \theta) & 0 \end{pmatrix}$. By Assumption 3 and 1 it can be seen that:

$$\begin{aligned} & |\det(\mathbb{H}_*)| = \\ & \left| \det(P_{*c}) \det \left(-\frac{\partial \Phi}{\partial u}(t, x, u_c, \theta) P_{*c}^{-1} \left(\frac{\partial \Phi}{\partial u}(t, x, u_c, \theta) \right)^T \right) \right| \\ & > 0, \end{aligned} \quad (\text{A.4})$$

and $\mathbb{H}_*$ is non-singular and there exists constants $\bar{\rho}_2 \geq \bar{\rho}_1 > 0$ such that $\bar{\rho}_1 I \leq \mathbb{H}_*^T \mathbb{H}_* \bar{\rho}_2 I$. Furthermore choose $u = 0$ and $\lambda = 0$, such that $|\frac{\partial L}{\partial u}(t, x, 0, 0, \theta)| \leq \varsigma_{\partial J}(|x|)$ and $|\frac{\partial L}{\partial \lambda}(t, x, 0, 0, \theta)| \leq G_\Phi(|x|)(1 + |\theta|) + \varsigma_k(|x|)$ from Assumption 1, 2 and 3.

Thus from (A.3) $\left| \begin{pmatrix} u^{*T} & \lambda^{*T} \end{pmatrix}^T \right| \leq \varsigma_{\mathcal{O}u\lambda} \left(\left| \begin{pmatrix} x^T & \theta^T \end{pmatrix}^T \right| \right)$

where $\varsigma_{\mathcal{O}u\lambda}(|s|) := \bar{\rho}_1^{-1} (G_\Phi(|s|)(1 + |s|) + \varsigma_k(|s|))^2 + \bar{\rho}_1^{-1} \varsigma_{\partial J}(|s|)^2$

B Proof of Proposition 2

The proof is divided into two parts: **i)** show that the integral of $\bar{f}_{\mathcal{O}_{u\lambda\bar{\theta}}}(t)^T \bar{f}_{\mathcal{O}_{u\lambda\bar{\theta}}}(t)$, where

$$\bar{f}_{\mathcal{O}_{u\lambda\bar{\theta}}}(t) := f_{\mathcal{O}_{u\lambda\bar{\theta}}}(t, x(t), u(t), \lambda(t), \epsilon(t), \tilde{\theta}(t)),$$

is bounded, by using the PE property; and **ii)** use the integral bound to prove UGA by contradiction. The proof is based on ideas from (Pantely, Loria & Teel 2001) and (Teel et al. 2002).

i) First we establish a bound on $\dot{\Phi}_g(t)$. We have

$$\begin{aligned} \dot{\Phi}_g(t) &= \frac{\partial g_x(t, x)}{\partial t} \Phi_\theta(t, x, u) + g_x(t, x) \frac{\partial \Phi_\theta(t, x, u)}{\partial t} \\ &+ \left(\frac{\partial g_x(t, x)}{\partial x} \Phi_\theta(t, x, u) + g_x(t, x) \frac{\partial \Phi_\theta(t, x, u)}{\partial x} \right) \dot{x} \\ &+ g_x(t, x) \frac{\partial \Phi_\theta(t, x, u)}{\partial u} \dot{u}, \end{aligned}$$

thus from Assumption 1, system (28) and update-law (30), there exists a bound $|\dot{\Phi}_g(t)| \leq \varsigma_{1x\dot{\Phi}_g}(|x|) + \varsigma_{1x\dot{\Phi}_g}(|x|)\varsigma_{u\dot{\Phi}_g}(|z_{u\lambda\bar{\theta}}|_{\mathcal{O}_{u\lambda\bar{\theta}}})$ where $\varsigma_{1x\dot{\Phi}_g}, \varsigma_{1x\dot{\Phi}_g}, \varsigma_{1x\dot{\Phi}_g} : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ are continuous functions. This can be seen by following the same approach as in Lemma 3. From the assumption that x is uniformly bounded, we use $|x| \leq B_x$. The integrability of $\tilde{\theta}^T \tilde{\theta}$ is investigated by considering the auxiliary function: $V_{\theta aux1} := -\epsilon^T \Phi_g(t) \tilde{\theta}$, bounded by $B_{V_{\theta aux1}}(B_x, r) \geq V_{\theta aux1}$ where $r \geq |z_{u\lambda\bar{\theta}}|_{\mathcal{O}_{u\lambda\bar{\theta}}}$ and $r > 0$. Its derivative along the solutions of (27) and (31) is given by

$$\begin{aligned} \dot{V}_{\theta aux1} &= -\tilde{\theta}^T \Phi_g(t)^T \Phi_g(t) \tilde{\theta} + \epsilon^T \left(A^T \Phi_g(t) - \dot{\Phi}_g(t) \right) \tilde{\theta} \\ &+ \epsilon^T \Phi_g(t) \Gamma_{\tilde{\theta}}^{-1} \Phi_g(t)^T \epsilon \end{aligned} \quad (\text{B.1})$$

$$+ \epsilon^T \Phi_g(t) \Gamma_{\tilde{\theta}}^{-1} \Phi_g(t)^T \frac{\partial_x^2 \bar{L}_{\tilde{\theta}}}{\partial u \lambda}(t) \frac{\partial \bar{L}_{\tilde{\theta}}}{\partial u \lambda}(t), \quad (\text{B.2})$$

where $\frac{\partial_x^2 \bar{L}_{\tilde{\theta}}}{\partial u \lambda}(t) := \left(\left(\frac{\partial^2 L_{\tilde{\theta}}}{\partial x \partial u} \right)^T, \left(\frac{\partial^2 L_{\tilde{\theta}}}{\partial x \partial \lambda} \right)^T \right) \begin{pmatrix} t \\ x(t), u(t), \lambda(t), \hat{\theta}(t) \end{pmatrix}$

and $\frac{\partial \bar{L}_{\tilde{\theta}}}{\partial u \lambda}(t) := \left(\left(\frac{\partial L_{\tilde{\theta}}}{\partial u} \right)^T, \left(\frac{\partial L_{\tilde{\theta}}}{\partial \lambda} \right)^T \right) \begin{pmatrix} t \\ x(t), u(t), \lambda(t), \hat{\theta}(t) \end{pmatrix}$.

From Young's inequality, $na^T a + \frac{1}{n} b^T b \geq 2|b^T a|$, we have

$$\begin{aligned} & \epsilon^T \left(A^T \Phi_g(t) - \dot{\Phi}_g(t) \right) \tilde{\theta} \\ & \leq \frac{T}{\mu} \epsilon^T \left(A^T \Phi_g(t) - \dot{\Phi}_g(t) \right) \left(A^T \Phi_g(t) - \dot{\Phi}_g(t) \right)^T \epsilon \\ & + \frac{\mu}{4T} \tilde{\theta}^T \tilde{\theta}, \end{aligned} \quad (\text{B.3})$$

where $a = \tilde{\theta}$, $b = \epsilon^T \left(A^T \Phi_g(t) - \dot{\Phi}_g(t) \right)$ and $n = \frac{\mu}{2T}$ such that μ and T are positive scalars of desirable choice. Next

$$\begin{aligned} & \epsilon^T \Phi_g(t) \Gamma_{\tilde{\theta}}^{-1} \Phi_g(t)^T \frac{\partial_x^2 \bar{L}_{\tilde{\theta}}}{\partial u \lambda}(t) \frac{\partial \bar{L}_{\tilde{\theta}}}{\partial u \lambda}(t) \\ & \leq \epsilon^T \Phi_g(t) \Phi_g(t)^T \epsilon + \left(\frac{\partial \bar{L}_{\tilde{\theta}}}{\partial u \lambda}(t) \right)^T K_{L\Phi} \frac{\partial \bar{L}_{\tilde{\theta}}}{\partial u \lambda}(t) \end{aligned}$$

for $n = 1$, $a = \epsilon^T \Phi_g(t)$ and $b = \Gamma_{\hat{\theta}}^{-1} \Phi_g(t)^T \frac{\partial^2 \bar{L}_{\hat{\theta}}}{\partial u \lambda}(t) \frac{\partial \bar{L}_{\hat{\theta}}}{\partial u \lambda}(t)$, where $K_{L\Phi} := \frac{\partial^2 \bar{L}_{\hat{\theta}}}{\partial u \lambda}(t) \Phi_g(t) \Gamma_{\hat{\theta}}^{-T} \Gamma_{\hat{\theta}}^{-1} \Phi_g^T(t) \frac{\partial^2 \bar{L}_{\hat{\theta}}}{\partial u \lambda}(t)$. Hence

$$\begin{aligned} \dot{V}_{\theta_{aux1}} &\leq \frac{T}{\mu} \epsilon^T \left(A^T \Phi_g(t) - \dot{\Phi}_g(t) \right) \left(A^T \Phi_g(t) - \dot{\Phi}_g(t) \right)^T \epsilon \\ &\quad + \epsilon^T \Phi_g(t) \Gamma_{\hat{\theta}}^{-1} \Phi_g(t)^T \epsilon - \tilde{\theta}^T \Phi_g(t)^T \Phi_g(t) \tilde{\theta} \\ &\quad + \frac{\mu}{4T} \tilde{\theta}^T \tilde{\theta} + \epsilon^T \Phi_g(t) \Phi_g^T(t) \epsilon \\ &\quad + \frac{\partial \bar{L}_{\hat{\theta}}}{\partial u \lambda}(t) K_{L\Phi} \frac{\partial \bar{L}_{\hat{\theta}}}{\partial u \lambda}(t), \end{aligned} \quad (B.4)$$

and thus

$$\begin{aligned} &\tilde{\theta}(t)^T \left(\Phi_g(t)^T \Phi_g(t) - \frac{\mu}{4T} I \right) \tilde{\theta}(t) \\ &\leq \frac{T}{\mu} \epsilon^T \left(A^T \Phi_g(t) - \dot{\Phi}_g(t) \right) \left(A^T \Phi_g(t) - \dot{\Phi}_g(t) \right)^T \epsilon \\ &\quad + \epsilon^T \Phi_g(t) \left(\Gamma_{\hat{\theta}}^{-1} + I \right) \Phi_g^T(t) \epsilon \\ &\quad - \dot{V}_{\theta_{aux1}} + \frac{\partial \bar{L}_{\hat{\theta}}}{\partial u \lambda}(t) K_{L\Phi} \frac{\partial \bar{L}_{\hat{\theta}}}{\partial u \lambda}(t). \end{aligned} \quad (B.5)$$

The solution of (30) can be represented by: $\tilde{\theta}(\tau) = \tilde{\theta}(t) - R_{\Phi}(t, \tau)$, where $R_{\Phi}(t, \tau) := \Gamma_{\hat{\theta}}^{-1} \int_t^{\tau} \Phi_g^T(s) \left(\epsilon(s) + \frac{\partial^2 \bar{L}_{\hat{\theta}}}{\partial u \lambda}(s) \frac{\partial \bar{L}_{\hat{\theta}}}{\partial u \lambda}(s) \right) ds$, such that

$$\begin{aligned} &\tilde{\theta}(\tau)^T \left(\Phi_g(\tau)^T \Phi_g(\tau) - \frac{\mu}{4T} I \right) \tilde{\theta}(\tau) \\ &= \tilde{\theta}(t)^T \left(\Phi_g(\tau)^T \Phi_g(\tau) - \frac{\mu}{4T} I \right) \tilde{\theta}(t) \\ &\quad + R_{\Phi}(t, \tau)^T \left(\Phi_g(\tau)^T \Phi_g(\tau) - \frac{\mu}{4T} I \right) R_{\Phi}(t, \tau) \\ &\quad - 2\tilde{\theta}(t)^T \left(\Phi_g(\tau)^T \Phi_g(\tau) - \frac{\mu}{4T} I \right) R_{\Phi}(t, \tau). \end{aligned} \quad (B.6)$$

Again by Young's inequality, $-n_1 a_1^T a_1 - \frac{1}{n_1} b_1^T b_1 \leq -2|b_1^T a_1|$ and $-n_2 a_2^T a_2 - \frac{1}{n_2} b_2^T b_2 \leq 2|b_2^T a_2|$ with $a_1 = \Phi_g(\tau) \tilde{\theta}$, $a_2 = \sqrt{\frac{\mu}{4T}} \tilde{\theta}$, $b_1 = \Phi_g(\tau) R_{\Phi}(t, \tau)$, $b_2 = \sqrt{\frac{\mu}{4T}} R_{\Phi}(t, \tau)$ and $n_1 = n_2 = \frac{1}{\sigma}$, such that

$$\begin{aligned} &-2\tilde{\theta}(t)^T \Phi_g(\tau)^T \Phi_g(\tau) R_{\Phi}(t, \tau) \geq -\frac{1}{\sigma} \tilde{\theta}(t)^T \Phi_g(\tau)^T \Phi_g(\tau) \tilde{\theta}(t) \\ &\quad - \sigma R_{\Phi}(t, \tau)^T \Phi_g(\tau)^T \Phi_g(\tau) R_{\Phi}(t, \tau) \end{aligned}$$

and

$$\begin{aligned} &2 \frac{\mu}{4T} \tilde{\theta}(t)^T R_{\Phi}(t, \tau) \geq -\frac{1}{\sigma} \frac{\mu}{4T} \tilde{\theta}(t)^T \Phi_g(\tau)^T \Phi_g(\tau) \tilde{\theta}(t) \\ &\quad - \sigma \frac{\mu}{4T} R_{\Phi}(t, \tau)^T \Phi_g(\tau)^T \Phi_g(\tau) R_{\Phi}(t, \tau) \end{aligned}$$

which gives

$$\begin{aligned} &\tilde{\theta}(\tau)^T \left(\Phi_g(\tau)^T \Phi_g(\tau) - \frac{\mu}{4T} I \right) \tilde{\theta}(\tau) \\ &\geq (1-\sigma) R_{\Phi}(t, \tau)^T \left(\Phi_g(\tau)^T \Phi_g(\tau) - \frac{(1+\sigma)\mu}{(1-\sigma)4T} I \right) R_{\Phi}(t, \tau) \\ &\quad + \left(1 - \frac{1}{\sigma} \right) \tilde{\theta}(t)^T \left(\Phi_g(\tau)^T \Phi_g(\tau) - \frac{(1+\frac{1}{\sigma})\mu}{(1-\frac{1}{\sigma})4T} I \right) \tilde{\theta}(t) \end{aligned} \quad (B.7)$$

for $\sigma > 1$. From Holder's inequality,

$$\int |fg| \leq \left(\int |f|^2 \right)^{\frac{1}{2}} \left(\int |g|^2 \right)^{\frac{1}{2}}, \text{ with } g = 1 \text{ and } f = \Gamma_{\hat{\theta}}^{-1} \Phi_g^T(s) \epsilon(s) \text{ we get}$$

$$\left(\int_t^{\tau} \left| \Gamma_{\hat{\theta}}^{-1} \Phi_g^T(s) \epsilon(s) \right| ds \right)^2 \leq \int_t^{\tau} \left| \Gamma_{\hat{\theta}}^{-1} \Phi_g^T(s) \epsilon(s) \right|^2 ds (\tau - t)$$

and by similar arguments

$$\begin{aligned} &\left(\int_t^{\tau} \left| \Gamma_{\hat{\theta}}^{-1} \Phi_g^T(s) \frac{\partial^2 \bar{L}_{\hat{\theta}}}{\partial u \lambda}(s) \frac{\partial \bar{L}_{\hat{\theta}}}{\partial u \lambda}(s) \right| ds \right)^2 \\ &\leq \int_t^{\tau} \left| \Gamma_{\hat{\theta}}^{-1} \Phi_g^T(s) \frac{\partial^2 \bar{L}_{\hat{\theta}}}{\partial u \lambda}(s) \frac{\partial \bar{L}_{\hat{\theta}}}{\partial u \lambda}(s) \right|^2 ds (\tau - t), \end{aligned}$$

such that

$$\begin{aligned} &R_{\Phi}(t, \tau)^T R_{\Phi}(t, \tau) \\ &\leq 2 \int_t^{\tau} \epsilon(s)^T \Phi_g(s) \Gamma_{\hat{\theta}}^{-T} \Gamma_{\hat{\theta}}^{-1} \Phi_g^T(s) \epsilon(s) ds (\tau - t) \\ &\quad + 2 \int_t^{\tau} \frac{\partial \bar{L}_{\hat{\theta}}}{\partial u \lambda}(s) K_{L\Phi} \frac{\partial \bar{L}_{\hat{\theta}}}{\partial u \lambda}(s) ds (\tau - t). \end{aligned}$$

Let $B_{\Phi T}(B_x, r, T) := \lambda_{\max} \left(\Phi_g(t)^T \Phi_g(t) - \frac{(1+\sigma)\mu}{(1-\sigma)4T} I \right)$ be the maximal eigenvalue of $\left(\Phi_g(\tau)^T \Phi_g(\tau) - \frac{(1+\sigma)\mu}{(1-\sigma)4T} I \right)$ and $\lambda_{\max} K_{L\Phi}$ be the maximal eigenvalue of $K_{L\Phi}$, then from combining (B.5) and (B.7) and investigating the integral over τ , we get

$$\begin{aligned} &\left(1 - \frac{1}{\sigma} \right) \tilde{\theta}(t)^T \int_t^{t+T} \left(\Phi_g(\tau)^T \Phi_g(\tau) - \frac{(1+\frac{1}{\sigma})\mu}{(1-\frac{1}{\sigma})4T} I \right) d\tau \tilde{\theta}(t) \\ &\leq B_{\Phi T} \int_t^{t+T} \left(\int_t^{\tau} \epsilon(s)^T \Phi_g(s) \Phi_g^T(s) \epsilon(s) ds \right) (\tau - t) d\tau \\ &\quad + B_{\Phi T} \lambda_{\max} K_{L\Phi} \int_t^{t+T} \left(\int_t^{\tau} \frac{\partial \bar{L}_{\hat{\theta}}}{\partial u \lambda}(s)^T \frac{\partial \bar{L}_{\hat{\theta}}}{\partial u \lambda}(s) ds \right) (\tau - t) d\tau \\ &\quad + V_{\theta_{aux1}}(t) - V_{\theta_{aux1}}(t+T) \\ &\quad + \lambda_{\max} K_{L\Phi} \int_t^{t+T} \left(\frac{\partial \bar{L}_{\hat{\theta}}}{\partial u \lambda}(s) \frac{\partial \bar{L}_{\hat{\theta}}}{\partial u \lambda}(s) \right) ds \\ &\quad + \int_t^{t+T} \epsilon(\tau)^T \Phi_g(\tau) \left(\Gamma_{\hat{\theta}}^{-1} + I \right) \Phi_g^T(\tau) \epsilon(\tau) d\tau \\ &\quad + \frac{T}{\mu} \int_t^{t+T} \epsilon(\tau)^T \left(A^T \Phi_g(\tau) - \dot{\Phi}_g(\tau) \right) \left(A^T \Phi_g(\tau) - \dot{\Phi}_g(\tau) \right)^T \epsilon(\tau) d\tau. \end{aligned}$$

Further

$$\begin{aligned} & \int_t^{t+T} \left(\int_t^\tau \epsilon(s)^T \Phi_g(s) \Gamma_{\hat{\theta}}^{-T} \Gamma_{\hat{\theta}}^{-1} \Phi_g^T(s) \epsilon(s) ds \right) (\tau - t) d\tau \\ & \leq T^2 \int_t^{t+T} \epsilon(s)^T \Phi_g(s) \Gamma_{\hat{\theta}}^{-T} \Gamma_{\hat{\theta}}^{-1} \Phi_g^T(s) \epsilon(s) ds, \end{aligned}$$

such that for

$$\begin{aligned} & B_\epsilon(B_x, r, T) \\ & := \lambda_{\max} \left(\left(A^T \Phi_g(t) - \dot{\Phi}_g(t) \right) \left(A^T \Phi_g(t) - \dot{\Phi}_g(t) \right)^T + \right. \\ & \quad \left. \Phi_g(t) \left((B_{\Phi T} T^2 + 1) \Gamma_{\hat{\theta}}^{-1} + I \right) \Phi_g(t)^T \right) \end{aligned}$$

it follows that

$$\begin{aligned} & \left(1 - \frac{1}{\sigma} \right) \tilde{\theta}(t)^T \int_t^{t+T} \left(\Phi_g(\tau)^T \Phi_g(\tau) - \frac{(1 + \frac{1}{\sigma}) \mu}{(1 - \frac{1}{\sigma}) 4T} I \right) d\tau \tilde{\theta}(t) \\ & \leq V_{\theta a u x 1}(t) - V_{\theta a u x 1}(t + T) \\ & + B_V \int_t^{t+T} \left(\left(\frac{\partial \bar{L}_{\hat{\theta}}}{\partial u \lambda}(\tau) \right)^T \mathbb{H}_{\hat{\theta}}^T(\tau) \Gamma_{\mathbb{H}_{\hat{\theta}}}(\tau) \frac{\partial \bar{L}_{\hat{\theta}}}{\partial u \lambda}(\tau) + \epsilon(\tau)^T A \epsilon(\tau) \right) d\tau \end{aligned}$$

and

$$\begin{aligned} & \tilde{\theta}(t)^T \int_t^{t+T} \left(\Phi_g(\tau)^T \Phi_g(\tau) - \frac{(1 + \frac{1}{\sigma}) \mu}{(1 - \frac{1}{\sigma}) 4T} I \right) d\tau \tilde{\theta}(t) \\ & \leq \frac{1}{(1 - \frac{1}{\sigma})} (V_{\theta a u x 1}(t) - V_{\theta a u x 1}(t + T)) \\ & + B_V (V_{u \lambda \hat{\theta}}(T + t) - V_{u \lambda \hat{\theta}}(t)), \end{aligned}$$

where $B_V(B_x, r, T) := \frac{\lambda_{\max} B_\epsilon(t, T) + \lambda_{\max} K_{L\Phi}(B_{\Phi T} T^2 + 1)}{(1 - \frac{1}{\sigma}) \lambda_{\min} A}$.

By the PE assumption we can choose $T, \gamma, \mu > 0$ and $\sigma > 1$ such that, $B_\gamma < \gamma - \frac{(1 + \frac{1}{\sigma}) \mu}{(1 - \frac{1}{\sigma}) 4}$, where $B_\gamma > 0$ i.e.

$$\begin{aligned} & B_\gamma \tilde{\theta}(t)^T \tilde{\theta}(t) \\ & \leq \tilde{\theta}(t)^T \int_t^{t+T} \left(\Phi_g(\tau)^T \Phi_g(\tau) - \frac{(1 + \frac{1}{\sigma}) \mu}{(1 - \frac{1}{\sigma}) 4T} I \right) d\tau \tilde{\theta}(t) \\ & \leq \frac{1}{(1 - \frac{1}{\sigma})} V_{\theta a u x 1}(t) - V_{\theta a u x 1}(t + T) \\ & + B_V (V_{u \lambda \hat{\theta}}(T + t) - V_{u \lambda \hat{\theta}}(t)), \end{aligned}$$

$$\begin{aligned} & \int_{t_0}^{t_0+T_T} (V_{\theta a u x 1}(\tau) - V_{\theta a u x 1}(\tau + T)) d\tau \\ & = \int_{t_0}^{t_0+T} V_{\theta a u x 1}(\tau) d\tau - \int_{t_0+T_T}^{t_0+T_T+T} V_{\theta a u x 1}(\tau) d\tau \\ & \leq 2T B_V \theta a u x 1(B_x, r) \end{aligned}$$

and

$$\int_{t_0}^{t_0+T_T} \tilde{\theta}(\tau)^T \tilde{\theta}(\tau) d\tau \leq \frac{2T}{B_\gamma} \left(\frac{B_V \theta a u x 1(B_x, r)}{(1 - \frac{1}{\sigma})} + B_V V_{u \lambda \hat{\theta}}(t_0) \right).$$

Thus $\tilde{\theta}(\tau)^T \tilde{\theta}(\tau)$ is integrable. Let $\lambda_{\min} V$ be the minimum eigenvalue of $(1, \mathbb{H}_{\hat{\theta}}^T \Gamma_{\mathbb{H}_{\hat{\theta}}}, A)$, then there exists a constant $B_{f_{\mathcal{O}_{u \lambda \hat{\theta}}}}(B_x, r, T) > 0$, such that

$$\begin{aligned} & \int_{t_0}^{t_0+T_T} \bar{f}_{\mathcal{O}_{u \lambda \hat{\theta}}}(\tau)^T \bar{f}_{\mathcal{O}_{u \lambda \hat{\theta}}}(\tau) d\tau \\ & \leq \lim_{t \rightarrow \infty} \int_{t_0}^t \bar{f}_{\mathcal{O}_{u \lambda \hat{\theta}}}(\tau)^T \bar{f}_{\mathcal{O}_{u \lambda \hat{\theta}}}(\tau) d\tau \\ & \leq \frac{1}{\lambda_{\min} V} (V_{u \lambda \hat{\theta}}(t_0) - V_{u \lambda \hat{\theta}}(\infty)) \\ & \quad + \frac{2T}{B_\gamma} \left(\frac{B_V \theta a u x 1(B_x, r)}{(1 - \frac{1}{\sigma})} + B_V V_{u \lambda \hat{\theta}}(t_0) \right) \\ & \leq B_{f_{\mathcal{O}_{u \lambda \hat{\theta}}}}, \end{aligned} \tag{B.8}$$

and the integral of $\bar{f}_{\mathcal{O}_{u \lambda \hat{\theta}}}(t)^T \bar{f}_{\mathcal{O}_{u \lambda \hat{\theta}}}(t)$ is bounded.

ii) From The UGS property we have

$$|z_{u \lambda \hat{\theta}}(t)|_{\mathcal{O}_{u \lambda \hat{\theta}}} \leq \rho(|z_{u \lambda \hat{\theta} 0}|_{\mathcal{O}_{u \lambda \hat{\theta}}}), \quad \forall t \geq t_0$$

where $\rho \in \mathcal{K}_\infty$. Fix $r > 0, \varepsilon > 0$. Define $\Omega := \rho(r)$ and $\omega := \min \{\Omega, \rho^{-1}(\varepsilon)\}$ and note that $B_{f_{\mathcal{O}_{u \lambda \hat{\theta}}}}$ is given by ω and Ω . Define $T_T = \frac{2B_{f_{\mathcal{O}_{u \lambda \hat{\theta}}}}}{\omega_1}$, where ω_1 is specified later, and assume that for all $|z_{u \lambda \hat{\theta} 0}|_{\mathcal{O}_{u \lambda \hat{\theta}}} \leq r$, there exists $t' \in [t_0, T_T]$ such that $|z_{u \lambda \hat{\theta}}(t', z_{u \lambda \hat{\theta} 0})|_{\mathcal{A}} \leq \rho^{-1}(\varepsilon)$. Thus $|z_{u \lambda \hat{\theta}}(t, z_{u \lambda \hat{\theta} 0})|_{\mathcal{O}_{u \lambda \hat{\theta}}} \leq \rho(|z_{u \lambda \hat{\theta}}(t', z_{u \lambda \hat{\theta} 0})|_{\mathcal{O}_{u \lambda \hat{\theta}}}) \leq \rho(\rho^{-1}(\varepsilon)) = \varepsilon$ for $|z_{u \lambda \hat{\theta} 0}|_{\mathcal{O}_{u \lambda \hat{\theta}}} \leq r$ and $t \geq T_T + t_0$, which satisfies definition 3. Suppose the assumption is not true, i.e., there exists $|z_{u \lambda \hat{\theta} 0}|_{\mathcal{O}_{u \lambda \hat{\theta}}} \leq r$ such that $|z_{u \lambda \hat{\theta}}(t', z_{u \lambda \hat{\theta} 0})|_{\mathcal{O}_{u \lambda \hat{\theta}}} > \rho^{-1}(\varepsilon) \forall t' \in [t_0, T_T]$. Thus

$$\omega \leq |z_{u \lambda \hat{\theta}}(t', z_{u \lambda \hat{\theta} 0})|_{\mathcal{O}_{u \lambda \hat{\theta}}} \leq \Omega, \quad \forall t' \in [t_0, T_T]$$

which from radial unboundedness of $V_{u \lambda \hat{\theta}}$, related to (33), imply that there exist positive constants ω_1 , and Ω_1 such that

$$\omega_1 \leq \bar{f}_{\mathcal{O}_{u \lambda \hat{\theta}}}(t')^T \bar{f}_{\mathcal{O}_{u \lambda \hat{\theta}}}(t') \leq \Omega_1, \quad \forall t' \in [t_0, T_T]$$

Then

$$\begin{aligned} & \int_{t_0}^{t_0+T_T} \bar{f}_{\mathcal{O}_{u \lambda \hat{\theta}}}(\tau)^T \bar{f}_{\mathcal{O}_{u \lambda \hat{\theta}}}(\tau) d\tau \geq \int_{t_0}^{t_0+T_T} \omega_1 d\tau \\ & = \omega_1 T_T \\ & = 2B_{f_{\mathcal{O}_{u \lambda \hat{\theta}}}}, \end{aligned}$$

which contradicts (B.8), and the proposition is proved.