A Computational Approach to Explicit Feedback Stochastic Nonlinear Model Predictive Control

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Abstract—Nonlinear Model Predictive Control (NMPC) involves the solution at each sampling instant of a finite horizon optimal control problem subject to nonlinear system dynamics, and state and input constraints. Mathematical models of engineering systems usually contain some amount of uncertainty. In the robust NMPC problem formulation, the model uncertainty is taken into account. This paper presents an approximate multi-parametric Nonlinear Programming approach to explicit solution of feedback stochastic MPC problems for constrained nonlinear systems in the presence of stochastic uncertainty. It is assumed that the discrete probability distribution of the uncertainty is known. The mathematical expectation of the cost function is minimized subject to state and input constraints. The approximate explicit approach constructs a piecewise nonlinear approximation to the optimal sequence of feedback control policies. It is demonstrated by explicit feedback stochastic NMPC for a cart moving on a plane and attached to the wall via a spring.

I. INTRODUCTION

NONLINEAR Model Predictive Control (NMPC) involves the solution at each sampling instant of a finite horizon optimal control problem subject to nonlinear system dynamics and state and input constraints [1], [2], [3]. A survey of the numerical methods for on-line solution of NMPC problems is given in [4]. Most recently, an advanced-step NMPC controller with reduced on-line computational delay has been proposed in [5]. Several approaches to explicit solution of NMPC problems have been suggested in the literature. The benefits of an explicit solution, in addition to the efficient on-line computations, include also verifiability of the implementation. In [6], [7], [8], approaches for off-line computation of explicit suboptimal piecewise linear (PWL) predictive controllers for nonlinear systems with state and input constraints have been developed, based on the multi-parametric Nonlinear Programming (mp-NLP) ideas [9]. In [8], computational methods to handle non-convex mp-NLP problems have been suggested. Algorithms for solving mp-NLP problems, including the non-convex case, are described also in [10].

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Mathematical models of engineering systems usually contain some amount of uncertainty (typically unknown additive disturbances and/or uncertain model parameters). In the robust MPC problem formulation, the model uncertainty is taken into account. In some applications, the system to be controlled is described by a stochastic model where the probabilistic distribution of the uncertainty is assumed to be known. Several approaches for constrained *open-loop* MPC based on stochastic models (stochastic MPC) are proposed in [11]–[22]. The approaches [11], [12], [13] are based on linear state space models with stochastic parameters and/or additive noise and they optimize the expected value of the cost function subject to hard input constraints [11] or probabilistic constraints [12], [13]. In [14], [15], [16], [17], stochastic linear MPC approaches incorporating a probabilistic cost and probabilistic constraints are developed. Methods for *open-loop* stochastic MPC for nonlinear systems have been proposed in [18], [19]. The stochastic MPC approaches [11]–[19] are based on parametric probabilistic models. Alternatively, the stochastic systems can be modeled with non-parametric models, like Gaussian process models. An on-line optimization approach and an approximate explicit approach to *open-loop* stochastic NMPC based on Gaussian process models have been proposed in [20], [21] and in [22], respectively.

The stochastic MPC methods mentioned above employ an *open-loop* formulation, which guarantees the robust stability and the robust feasibility of the system, but it may be conservative. This is related to the fact that the control sequence has to ensure constraints fulfillment for all possible uncertainty scenarios without considering the fact that future measurements of the state contain information about past uncertain values. Similar to the *closed-loop* min-max NMPC approaches [23], [24], [25], the conservativeness of the *open-loop* stochastic NMPC can be overcome by a *closed-loop* stochastic NMPC formulation, where the optimization is performed over a sequence of feedback control policies. In [26], [27], methods for *closed-loop* stochastic NMPC based on on-line optimization have been proposed.

Based on the related min-max approximate mp-NLP approach in [28] we consider here the explicit solution of *closed-loop (feedback) stochastic* NMPC problems for constrained nonlinear systems in the presence of stochastic uncertainty. It is assumed that the discrete probability distribution of the uncertainty is known. The approach constructs a piecewise nonlinear (PWNL) approximation to

the optimal sequence of feedback control policies for efficient on-line implementation.

II. FORMULATION OF THE FEEDBACK STOCHASTIC NMPC PROBLEM AS AN MP-NLP PROBLEM

Consider the discrete-time nonlinear system:

$$\begin{aligned} x(t+1) &= f(x(t), u(t), w(t)) \\ y(t) &= h(x(t), u(t), w(t)), \end{aligned} \quad (1)$$

where $x(t) \in \mathfrak{R}^n$, $u(t) \in \mathfrak{R}^m$, $y(t) \in \mathfrak{R}^s$ and $w(t) \in \mathfrak{R}^n$ are the state, input, output and uncertainty variables, and $t \in \mathbb{Z}_{\geq 0}$ is the discrete time. The input and the output variables are required to fulfill the following constraints:

$$u_{\min} \leq u \leq u_{\max}, \quad y_{\min} \leq y \leq y_{\max}. \quad (2)$$

The following assumptions are made:

A1. f and h are C^2 functions with $f(0,0,0) = 0$, $h(0,0,0) = 0$.

A2. The uncertainty $w(t) = [w_1(t), w_2(t), \dots, w_n(t)]$ includes both internal (state-dependent) and external uncertainty, i.e. it has the form $w_i(t) = \lambda_i x_i(t) + \gamma_i(t)$, where $x_i(t)$ is the i -th element of the state vector $x(t)$. Here λ_i is a discrete random parameter, associated to the internal (model) uncertainty. It takes values in the set $\Lambda_i = \{\lambda_i^1, \lambda_i^2, \dots, \lambda_i^{n_{\lambda_i}}\}$ and is characterized with the probability mass function $\varphi_i : \Lambda_i \rightarrow [0, 1]$:

$$\varphi_i(\lambda_i^j) = \Pr(\lambda_i = \lambda_i^j), \quad j = 1, 2, \dots, n_{\lambda_i} \quad \text{with} \quad \sum_{j=1}^{n_{\lambda_i}} \varphi_i(\lambda_i^j) = 1. \quad (3)$$

The external uncertainty $\gamma_i(t)$ is stochastic and piecewise constant with infrequent changes in sense that $\gamma_i(t) = \text{const}$ for periods of time, which are not less than N_γ ($N_\gamma \in \mathbb{N}$ is supposed to be sufficiently large). It is assumed that γ_i takes values in the set $\Gamma_i = \{\gamma_i^1, \gamma_i^2, \dots, \gamma_i^{n_{\gamma_i}}\}$, which contains the nominal value $\gamma_i = 0$, and it is characterized with the probability mass function $\psi_i : \Gamma_i \rightarrow [0, 1]$:

$$\psi_i(\gamma_i^j) = \Pr(\gamma_i = \gamma_i^j), \quad j = 1, 2, \dots, n_{\gamma_i} \quad \text{with} \quad \sum_{j=1}^{n_{\gamma_i}} \psi_i(\gamma_i^j) = 1. \quad (4)$$

The overall vector of uncertain model parameters is denoted $\lambda = [\lambda_1, \lambda_2, \dots, \lambda_n] \in \Lambda$ with $\Lambda = \Lambda_1 \times \Lambda_2 \times \dots \times \Lambda_n$. Given $\lambda^j = [\lambda_1^{i_1}, \lambda_2^{i_2}, \dots, \lambda_n^{i_n}]$, $i_1 \in \{1, 2, \dots, n_{\lambda_1}\}$, $\dots$, $i_n \in \{1, 2, \dots, n_{\lambda_n}\}$, the probability mass function $\varphi : \Lambda \rightarrow [0, 1]$ is:

$$\varphi(\lambda^j) = \Pr(\lambda = \lambda^j) = \varphi_1(\lambda_1^{i_1}) \varphi_2(\lambda_2^{i_2}) \dots \varphi_n(\lambda_n^{i_n}) \quad (5)$$

with $j = 1, \dots, n_\lambda$ and $n_\lambda = n_{\lambda_1} n_{\lambda_2} \dots n_{\lambda_n}$. Similarly, the overall vector of external uncertainty is denoted

$\gamma = [\gamma_1, \gamma_2, \dots, \gamma_n] \in \Gamma$ with $\Gamma = \Gamma_1 \times \Gamma_2 \times \dots \times \Gamma_n$. Given $\gamma^j = [\gamma_1^{i_1}, \gamma_2^{i_2}, \dots, \gamma_n^{i_n}]$, $i_1 \in \{1, 2, \dots, n_{\gamma_1}\}$, $\dots$, $i_n \in \{1, 2, \dots, n_{\gamma_n}\}$, the probability mass function $\psi : \Gamma \rightarrow [0, 1]$ is:

$$\psi(\gamma^j) = \Pr(\gamma = \gamma^j) = \psi_1(\gamma_1^{i_1}) \psi_2(\gamma_2^{i_2}) \dots \psi_n(\gamma_n^{i_n}) \quad (6)$$

with $j = 1, \dots, n_\gamma$ and $n_\gamma = n_{\gamma_1} n_{\gamma_2} \dots n_{\gamma_n}$.

Then, the overall uncertainty is $w = \text{diag}(\lambda)x + \gamma$.

A3. $y_{\min} < 0 < y_{\max}$ and $u_{\min} < 0 < u_{\max}$.

As in [23], first a H_∞ control problem is defined:

Problem P1 (H_∞ control problem):

Design a state-feedback control law:

$$u = k(x) \quad (7)$$

guaranteeing that the closed-loop system (1)–(7) with input $w = \text{diag}(\lambda)x + \gamma$, $\lambda \in \Lambda$, $\gamma \in \Gamma$, and output y has a finite l_2 -gain $\leq \alpha$ in a bounded positively invariant set Ω , that is, $\forall x(t) \in \Omega$:

i. $x(t+i) \in \Omega$, $\forall i > 0$.

ii. $u_{\min} \leq k(x(t+i)) \leq u_{\max}$ and

$y_{\min} \leq h(x(t+i), k(x(t+i)), w(t+i)) \leq y_{\max}$, $\forall i \geq 0$.

iii. There exists a positive definite function $\beta(x(t))$, such that $\forall T \geq 0$:

$$\sum_{i=0}^T \|y(t+i)\|^2 \leq \alpha^2 \sum_{i=0}^T \|w(t+i)\|^2 + \beta(x(t)) \quad (8)$$

for any non-zero w .

The following assumption is also made [23]:

A4. There exists an auxiliary control law $u = k_a(x)$ that solves the H_∞ control problem, with a domain of attraction $\Omega_a(k_a, \alpha, \alpha_\Delta)$, whose boundary is assumed to be a level curve of a positive function $V_{k_a}(x)$ such that:

$$V_{k_a}(f(x, k_a(x), w)) - V_{k_a}(x) < -\frac{1}{2}(\|y\|^2 - \alpha^2 \|w\|^2), \quad (9)$$

$$\forall x \in \Omega_a(k_a, \alpha, \alpha_\Delta), \quad \forall \lambda \in \Lambda, \quad \forall \gamma \in \Gamma$$

and $V_{k_a}(0) = 0$.

Denote with:

$$\begin{aligned} K &= \{k_0, k_1, \dots, k_{N-1}\} \\ &\triangleq \{k_0(x_{t|t}), k_1(x_{t+1|t}), \dots, k_{N-1}(x_{t+N-1|t})\} \end{aligned} \quad (10)$$

a vector of feedback control policies. It is supposed that a full measurement x of the state is available at the current time t . We consider the following feedback stochastic NMPC problem:

Problem P2 (Constrained feedback stochastic NMPC problem):

Suppose that assumptions A1–A4 hold. For the current x , the feedback stochastic NMPC solves the following optimization problem:

$$V_E^*(x) = \min_K \min_{\lambda \in \Lambda, \gamma \in \Gamma} E \{J(K, x, \lambda, \gamma)\} \quad (11)$$

subject to $x_{t|t} = x$ and:

$$y_{\min} \leq y_{t+i|t} \leq y_{\max}, \forall \lambda \in \Lambda, \forall \gamma \in \Gamma, i = 1, \dots, N \quad (12)$$

$$u_{\min} \leq u_{t+i} \leq u_{\max}, i = 0, 1, \dots, N-1 \quad (13)$$

$$x_{t+N|t} \in \Omega_a(k_a, \alpha, \alpha_\Delta), \forall \lambda \in \Lambda, \forall \gamma \in \Gamma \quad (14)$$

$$u_{t+i} = k_i(x_{t+i|t}), i = 0, 1, \dots, N-1 \quad (15)$$

$$x_{t+i+1|t} = f(x_{t+i|t}, u_{t+i}, w_{t+i}), i \geq 0 \quad (16)$$

$$y_{t+i|t} = h(x_{t+i|t}, u_{t+i}, w_{t+i}), i \geq 0 \quad (17)$$

and the cost function given by:

$$J(K, x, \lambda, \gamma) = \frac{1}{2} \sum_{i=0}^{N-1} \left[\|y_{t+i|t}\|^2 - \alpha^2 \|w_{t+i}\|^2 \right] + V_{k_a}(x_{t+N|t}). \quad (18)$$

Here, N is a finite horizon, α is the l_2 -gain which is interpreted as the uncertainty attenuation level, and $E\{\cdot\}$ means mathematical expectation. It is supposed that $N \ll N_\gamma$, and by Assumption A2 it can be accepted that $\gamma_{t+i} = \text{const}$, $i = 0, 1, \dots, N-1$. Then by Assumption A2, the expectation can be expressed:

$$E_{\lambda \in \Lambda, \gamma \in \Gamma} \{J(K, x, \lambda, \gamma)\} = \sum_{i=1}^{n_\lambda} \sum_{j=1}^{n_\gamma} J(K, x, \lambda^i, \gamma^j) \varphi(\lambda^i) \psi(\gamma^j) \quad (19)$$

An auxiliary control law $k_a(x)$ is typically obtained by solving the H_∞ control problem for the linearized system [23]. Thus, a practical way to compute a nonlinear control $k_a(x)$ satisfying assumption A4 for nonlinear input-affine systems is suggested in [23].

An optimal solution to the feedback stochastic NMPC problem (11)–(18) is denoted $K^* = \{k_0^*, k_1^*, \dots, k_{N-1}^*\}$ and the control input is chosen according to the receding horizon policy $u(x_{t|t}) = k_0^*(x_{t|t})$.

The following assumption is made:

A5. Each feedback control policy $k_i(x_{t+i|t})$, $i = 0, \dots, N-1$ has the form:

$$k_i(x_{t+i|t}) = \eta_i k_a(x_{t+i|t}) + r_i(\xi_i, x_{t+i|t}) = g_i(p_i, x_{t+i|t}), \quad (20)$$

where $p_i = [\eta_i^T \xi_i^T]^T \in \mathfrak{R}^{n_i}$ are the parameters that need to be optimized, $k_a(x_{t+i|t})$ is an auxiliary control law that satisfies assumption A4, and $r_i(\xi_i, x_{t+i|t})$ is a parameterized continuous function with $r_i(\xi_i, 0) = 0$.

In general, the parameterization of the form (20) would lead to an approximate solution to the feedback stochastic NMPC problem (11)–(18). Denote with P the whole set of parameters that need to be determined, i.e.:

$$P = [p_0^T \ p_1^T \ \dots \ p_{N-1}^T]^T \in \mathfrak{R}^{n_p}, \quad n_p = \sum_{i=0}^{N-1} n_i. \quad (21)$$

Then, the expected value (19) of the cost function is:

$$V_E(P, x) = E_{\lambda \in \Lambda, \gamma \in \Gamma} \{J(P, x, \lambda, \gamma)\}. \quad (22)$$

It should be noted that the argument K in the cost function (18) is now replaced with the argument P .

The optimization problem (11)–(18) can be formulated in a compact form as follows:

Problem P3:

$$V_E^*(x) = \min_P \sum_{i=1}^{n_\lambda} \sum_{j=1}^{n_\gamma} J(P, x, \lambda^i, \gamma^j) \varphi(\lambda^i) \psi(\gamma^j) \quad (23)$$

$$\text{subject to: } G(P, x, \lambda, \gamma) \leq 0, \quad \forall \lambda \in \Lambda, \forall \gamma \in \Gamma. \quad (24)$$

Problem P3 defines an mp-NLP, since it is NLP in P parameterized by x . We remark that the constraints function $G(P, x, \lambda, \gamma)$ in (24) is implicitly defined by (12)–(17). Also, since Λ and Γ are finite sets, (24) represents a finite number of constraints. It should be noted that the number of constraints (24) increases rapidly with the increase of the horizon and the sizes n_λ and n_γ of the uncertainty sets Λ and Γ . Thus, as the horizon increases from N_1 to N_2 and the sizes of the sets Λ and Γ increase from n_λ^1 to n_λ^2 , and from n_γ^1 to n_γ^2 , respectively, the number of constraints will increase $(N_2 n_\lambda^2 n_\gamma^2) / (N_1 n_\lambda^1 n_\gamma^1)$ times. This leads to a stronger motivation for an explicit approach, where the computational complexity is handled in off-line.

Define the set of N -step robustly feasible initial states:

$$X_f = \{x \in \mathfrak{R}^n \mid G(P, x, \lambda, \gamma) \leq 0, \forall \lambda \in \Lambda, \forall \gamma \in \Gamma \text{ for some } P \in \mathfrak{R}^{n_p}\}. \quad (25)$$

In parametric programming problems one seeks the solution $P^*(x)$ as an explicit function of the parameters x in some set $X \subseteq X_f \subseteq \mathfrak{R}^n$ [9].

III. APPROXIMATE MP-NLP APPROACH TO EXPLICIT FEEDBACK STOCHASTIC NMPC

We restrict our attention to a hyper-rectangle $X \subset \mathfrak{R}^n$ where we seek to approximate the optimal sequence of control policies $K^* = \{k_0^*, k_1^*, \dots, k_{N-1}^*\}$. The associated optimal control input is $u_{t+i}^* = k_i^*(x_{t+i|t}) = g_i(p_i^*, x_{t+i|t})$, $i = 0, 1, \dots, N-1$, where $P^* = [p_0^{*T} \ p_1^{*T} \ \dots \ p_{N-1}^{*T}]^T$ is determined by solving problem P3. We require that the state space partition is orthogonal and can be represented as a k -d tree. The main idea of the approximate mp-NLP approach is to construct a *piecewise nonlinear* (PWNL) approximation $\hat{K} = \{\hat{k}_0, \hat{k}_1, \dots, \hat{k}_{N-1}\}$ to the optimal feedback $K^* = \{k_0^*, k_1^*, \dots, k_{N-1}^*\}$ on X , where the constituent *nonlinear* control functions $K_{X_i} = \{k_{0,X_i}, k_{1,X_i}, \dots, k_{N-1,X_i}\}$ are defined on hyper-rectangles X_i covering X . Let $K_{X_0} = \{k_{0,X_0}, \dots, k_{N-1,X_0}\}$ be an approximation to the optimal solution $K^* = \{k_0^*, \dots, k_{N-1}^*\}$, valid in the whole

hyper-rectangle X_0 . Denote with $P_{X_0} = [p_{0,X_0}^T \dots p_{N-1,X_0}^T]^T$ the parameters of K_{X_0} . The corresponding approximate value of the control input is $\hat{u}_{t+i} = k_{i,X_0}(x_{t+i|t}) = g_i(p_{i,X_0}, x_{t+i|t})$, $i = 0, 1, \dots, N-1$. Let $\hat{V}_E(P_{X_0}, x)$ be the cost function value due to initial state $x = x_{i|t}$ and control function K_{X_0} , i.e. $\hat{V}_E(P_{X_0}, x) = \min_{\lambda \in \Lambda, \gamma \in \Gamma} \{J(P_{X_0}, x, \lambda, \gamma)\}$. Then, the approximate control function K_{X_0} is determined by applying the following procedure:

Procedure 1:

Suppose A1–A5 hold. Consider any hyper-rectangle $X_0 \subseteq X_f$ with a set of points $V_0 = \{v_0, v_1, v_2, \dots, v_{N_1}\}$. Compute the parameters $P_{X_0} = [p_{0,X_0}^T \dots p_{N-1,X_0}^T]^T$ of the control function $K_{X_0} = \{k_{0,X_0}, \dots, k_{N-1,X_0}\}$ by solving the following NLP:

$$\min_{P_{X_0}} \sum_{i=0}^{N_1} (\hat{V}_E(P_{X_0}, v_i) - V_E^*(v_i)) + \mu \|g_0(p_{0,X_0}, v_i) - g_0(p_0^*, v_i)\|^2 \quad (26)$$

subject to:

$$G(P_{X_0}, v_i, \lambda, \gamma) \leq 0, \forall v_i \in V_0, \forall \lambda \in \Lambda, \forall \gamma \in \Gamma. \quad (27)$$

We remark that the optimal parameters p_0^* in the feedback function $k_0^*(v_i) = g_0(p_0^*, v_i)$ in (26) are determined by solving problem P3 for $x = v_i$ and the parameter $\mu > 0$ is a weighting coefficient.

Note that the control function $K_{X_0} = \{k_{0,X_0}, \dots, k_{N-1,X_0}\}$, computed with Procedure 1, satisfies the constraints in problem P3 only for the discrete set of points V_0 in the hyper-rectangle X_0 and for the discrete sets of values Λ and Γ of the internal and external uncertainties.

Suppose that the parameter vector P_{X_0} of the control function K_{X_0} , valid in X_0 , has been determined by applying Procedure 1. Then, for the cost function approximation error in X_0 we have:

$$\varepsilon(x) = \hat{V}_E(P_{X_0}, x) - V_E^*(x) \leq \varepsilon_0, \quad x \in X_0. \quad (28)$$

An estimate $\hat{\varepsilon}_0$ of the error bound ε_0 is computed as:

$$\hat{\varepsilon}_0 = \max_{i \in \{0, 1, 2, \dots, N_1\}} (\hat{V}_E(P_{X_0}, v_i) - V_E^*(v_i)). \quad (29)$$

If $\hat{\varepsilon}_0 > \bar{\varepsilon}$, where $\bar{\varepsilon} > 0$ is the specified tolerance of the cost function approximation error, the region X_0 is divided and the procedure is repeated for the new regions. The detailed approximate mp-NLP algorithm is given in [28].

It should be noted that in case of non-convexity of

problem P3, it can not be guaranteed that the approximation error $\varepsilon(x)$ associated to the explicit feedback stochastic NMPC will satisfy the requirement $\varepsilon(x) \leq \bar{\varepsilon}$ for all $x \in X$. The non-convexity may also imply that the constraints (24) are violated at some points of the state space. In this respect, the described computational method does not necessarily lead to guaranteed properties, but when combined with verification and analysis methods gives a practical tool for development and implementation of explicit feedback stochastic NMPC. The possibility for implementation verification is a significant advantage of the explicit NMPC in comparison to NMPC based on real-time optimization.

IV. SIMULATION EXAMPLE

Consider a cart with a mass M moving on a plane [23]. The carriage is attached to the wall via a spring with elasticity $\rho = \rho_0 e^{-x_1}$, where x_1 is the displacement of the carriage from the equilibrium position associated with the external force $u = 0$ [23]. A damper with damping factor h_d affects the system in a resistive way. The damping factor h_d is uncertain, but it is known that $h_d = \bar{h}_d + \lambda$. Here, $\bar{h}_d = 1.1$ and λ is a stochastic parameter. It is supposed that $-0.5 \leq \lambda \leq 0.5$ and the following discrete set of values is considered $\lambda \in \Lambda = \{-0.5, 0, 0.5\}$ with the corresponding values of the probability mass function $\varphi(-0.5) = 0.2$, $\varphi(0) = 0.6$, $\varphi(0.5) = 0.2$. The external uncertainty for this system is $\gamma = 0$. The system is described by the following nonlinear discrete-time model [23]:

$$x_1(t+1) = x_1(t) + T_s x_2(t) \quad (30)$$

$$x_2(t+1) = x_2(t) - T_s \frac{\rho_0}{M} e^{-x_1(t)} x_1(t) - T_s \frac{\bar{h}_d}{M} x_2(t) + T_s \frac{u(t)}{M} + T_s w(t), \quad (31)$$

where x_2 is the carriage velocity, $w(t) = -\frac{\lambda}{M} x_2(t)$ is a state dependent (internal) uncertainty, $T_s = 0.4$ is the sampling time, $M = 1$ and $\rho_0 = 0.33$. The following input and state constraints are imposed on the system:

$$-4 \leq u \leq 4, \quad -1.3 \leq x_2 \leq 1.3. \quad (32)$$

The horizon is $N = 15$ and the terminal constraint is:

$$x_{t+N|t} \in \Omega_a, \quad \Omega_a = \{x \in \mathbb{R}^n | x^T \Sigma x \leq \delta\}, \quad (33)$$

where $\delta = 0.001$ [23] and $\Sigma = \begin{bmatrix} 1.3 & 1.9 \\ 1.9 & 3.0 \end{bmatrix}$.

The mp-NLP approach described in section III is applied to design an explicit feedback stochastic NMPC controller for the cart. The NMPC minimizes the mathematical expectation (19) of the cost function (18) subject to the system equations (30), (31) and the constraints (32)–(33). In (18), it is chosen $\alpha = 1$ and the terminal penalty is given by

$V_{k_a} = x^T \Sigma x$ [23]. Like in [23], the feedback functions $k_i(x_{t+i|t})$, $i = 0, \dots, N-1$ have the form:

$$k_i(p_i, x_{t+i|t}) = \eta_i k_a(x_{t+i|t}) + \xi_{i,1} x_{1,t+i|t}^2 + \xi_{i,2} x_{2,t+i|t}^2, \quad (34)$$

where $p_i = [\eta_i \ \xi_{i,1} \ \xi_{i,2}]^T$ are the parameters that need to be optimized and $k_a(x_{t+i|t})$ is the auxiliary control law. The expression (34) implies that for relatively small absolute deviations from the equilibrium (small $x_{1,t+i|t}^2$ and $x_{2,t+i|t}^2$) the control input value will be generated mainly by the auxiliary control law $k_a(x_{t+i|t})$. The control law $k_a(x_{t+i|t})$ is determined by applying the method in [23]:

$$k_a(x_{t+i|t}) = -[1 \ 0] R^{-1} \begin{bmatrix} F_2^T \\ F_3^T \end{bmatrix} \Sigma f_1(x_{t+i|t}), \quad (35)$$

where:

$$f_1(x_{t+i|t}) = \begin{bmatrix} x_{1,t+i|t} + T_s x_{2,t+i|t} \\ x_{2,t+i|t} - T_s \frac{\rho_0}{M} e^{-x_{1,t+i|t}} x_{1,t+i|t} - T_s \frac{\bar{h}_d}{M} x_{2,t+i|t} \end{bmatrix} \quad (36)$$

$$F_2 = \begin{bmatrix} 0 \\ T_s \\ M \end{bmatrix}, F_3 = \begin{bmatrix} 0 \\ T_s \end{bmatrix}, R = \begin{bmatrix} F_2^T \Sigma F_2 + I & F_2^T \Sigma F_3 \\ (F_2^T \Sigma F_3)^T & F_3^T \Sigma F_3 - \alpha^2 I \end{bmatrix} \quad (37)$$

In [7], a condition on the approximation tolerance has been derived such that the asymptotic stability of the nonlinear system in closed-loop with the approximate explicit NMPC is guaranteed. According to this condition, the tolerance is chosen to be dependent on the state, which would lead to a state space partition with less complexity in comparison to that corresponding to an uniform tolerance. Here, a similar approach is applied and the approximation tolerance is chosen to be dependent on the state as $\bar{\varepsilon}(X_0) = \max(\bar{\varepsilon}_a, \bar{\varepsilon}_r \min_{x \in X_0} V_E^*(x))$, where $\bar{\varepsilon}_a = 0.005$ and $\bar{\varepsilon}_r = 0.03$ are the absolute and the relative tolerances.

The state space partition of the explicit feedback stochastic NMPC controller is shown in Fig. 1.

Fig. 1. State space partition of the explicit feedback stochastic NMPC and the state trajectories corresponding to $\lambda = -0.5$, $\lambda = 0$, $\lambda = 0.5$.

The partition has 150 regions and 11 levels of search. Totally, 27 arithmetic operations are needed in real-time to compute the control input (11 comparisons, 10 multiplications, 5 additions and 1 exponential).

In Fig. 2, the suboptimal feedback function $\hat{u}(x_1, x_2) = \hat{k}_0(x_1, x_2)$ is shown.

Fig. 2. The suboptimal feedback function $\hat{u}(x_1, x_2) = \hat{k}_0(x_1, x_2)$ (view rotated on 140°).

The performance of the closed-loop system was simulated for initial state $x(0) = [4 \ 1.5]^T$ and for the three values of the stochastic parameter λ . The response is depicted in the state space (Fig. 1), as well as trajectories in time (Fig. 3 and Fig. 4). In Fig. 3 and Fig. 4, the control and state trajectories obtained with the explicit min-max NMPC controller [28] are given for comparison. The cost function values corresponding to the closed-loop trajectories associated to the explicit stochastic NMPC and to the explicit min-max NMPC are $\hat{V}_E = 121.64$ and $\hat{V}_{\min-\max} = 122.57$ (for $h_d = 0.6$), and $\hat{V}_E = 141.59$ and $\hat{V}_{\min-\max} = 141.70$ (for $h_d = 1.6$). Therefore, the explicit min-max NMPC is slightly more conservative, since it is characterized with larger values of the cost function in comparison to the explicit stochastic NMPC.

Fig. 3. Control input (left) and state trajectory (right) for $h_d = 0.6$.

Fig. 4. Control input (left) and state trajectory (right) for $h_d = 1.6$.

It can be seen that the explicit feedback stochastic NMPC controller brings the cart to the equilibrium despite of the presence of stochastic uncertainty, and the constraints imposed on the system are satisfied.

V. CONCLUSION

This paper presents an approximate mp-NLP approach to explicit solution of feedback stochastic MPC problems for constrained nonlinear systems in the presence of stochastic uncertainty. It is applied to design an explicit feedback stochastic NMPC for a cart moving on a plane and attached to the wall via a spring. Results show that the explicit stochastic NMPC controller brings the cart to the equilibrium despite of the presence of stochastic uncertainty, and the constraints imposed on the system are satisfied.

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