Inf-Sup Control of Discontinuous PWA Systems

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***Abstract*— Piecewise affine (PWA) systems is an important class of systems. In this paper we consider optimal control of constrained, discontinuous, discrete-time, piecewise affine systems with state and input dependent disturbances. We seek to pre-compute, via dynamic programming, an explicit control law for these systems when a piecewise affine cost function is utilized. The main difficulty with this problem class is that, even for initial states for which the value function of the optimal control problem is finite, there might not exist a control law that attains the infimum. Hence, we propose a method that is guaranteed to obtain a sub-optimal solution, and where the degree of sub-optimality can be *a priori* specified. This is achieved by approximating the underlying sub-problems with a piecewise parametric linear program.**

***Index Terms*— Optimal control. Piecewise affine systems. Parametric programming. Approximate solutions to inf-sup problems.**

I. INTRODUCTION

Methods for computing explicit control laws for discrete-time, constrained, piecewise affine (PWA) systems have been reported in the control literature recently [1]–[6]. Usually dynamic programming is utilized for this purpose [1], [2], [7], [8]. In [1] a dynamic programming approach for continuous PWA systems is proposed and in [2] continuous PWA systems subject to state- and input-dependent disturbances are considered. In this note we consider *discontinuous* PWA systems subject to input- and state-dependent disturbances and where the cost function is piecewise affine. Methods for computing explicit control laws for this problem class have not yet been reported in the literature.

In [1] discontinuous PWA systems are briefly mentioned. However, disturbances are not present and the topic is not treated in detail; for instance that an optimizer exists[1] cannot be guaranteed. In addition, the authors represent the domain of the PWA state update equation by closed polyhedra, as described in [9], and therefore small gaps are introduced in the domain of the state update equation. Consequently, from a theoretical point of view, the state trajectory may vanish. One can argue that the control scheme is to be implemented on a microchip or computer and therefore is subject to a finite arithmetic precision. In this paper we will look at the problem from a theoretical viewpoint, remove the need to introduce small gaps and outline the foundation for a complete computational procedure for computing explicit control laws for this problem class.

An alternative way of describing a discontinuous PWA system is to transform the state update equation into a difference inclusion by performing a regularization, see e.g. [10]. With this system description, the successor state may be set-valued for a given initial state, control input, and disturbance. We seek to avoid this situation by still treating the system as a difference equation.

In this paper we represent the domain of PWA systems by a union of a finite number of open, closed and/or neither open nor closed polyhedra. A solution to the optimal control problem may not exist in this case. However, solutions that render the cost function within a small neighborhood of the infimum/supremum are guaranteed to exist. We propose a procedure that obtains a sub-optimal solution to the optimal control problem when the solution does not exist, and the exact solution when it does. This approach does not introduce gaps in the domain of the state update equation, we do not assume that a solution to the optimal control problem exists, and the state update equation is not transformed into a difference inclusion, and thus, the dynamic programming approach is relatively simple from the theoretical point of view. In addition, the proposed procedure allows the degree of sub-optimality to be *a priori* specified.

PAPER STRUCTURE: Section II introduces basic notation and definitions. In Section III we introduce the basic building block in the paper, namely how to obtain ε-optimal solutions to parametric linear programs (pLP) with strict and non-strict inequality constraints. This building block is then used to obtain sub-optimal solutions to minimization of PWA functions in Section IV, to min-max problems in Section V, and finally it is demonstrated in Section VI how these procedures can be used in the dynamic programming approach for the purpose of obtaining explicit, sub-optimal, solutions to robust optimal control problems for discontinuous PWA systems.

II. PRELIMINARIES

A. Basic Notation and Fundamental Results

Recall the following; the *affine hull* of a set S is the intersection of all affine sets containing S, and is denoted $\mathrm{aff}(S)$. The *dimension of a set* $S \subset \mathbb{R}^n$ is the dimension of $\mathrm{aff}(S)$, and is denoted $\dim(S)$; if $\dim(S) = n$, then S is said to be full-dimensional. The *closure* of a set S is denoted $\mathrm{cl}(S)$. The *relative interior* of a set S is the interior relative to $\mathrm{aff}(S)$, i.e. $\mathrm{relint}(S) := \{x \in S \mid B(x,r) \cap \mathrm{aff}(S) \subseteq S \text{ for some } r > 0\}$, where the ball $B(x,r) := \{y \mid \|y-x\| \le r\}$ and $\|\cdot\|$ is any norm. We denote the orthogonal projection of a set $S \subseteq \mathbb{R}^n \times \mathbb{R}^m$ to the x-space by $\mathrm{Proj}_x\, S := \{x \in \mathbb{R}^n \mid \exists u \in \mathbb{R}^m : (x,u) \in S\}$.

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[1]Given an optimization problem, e.g. $J^* := \inf_{x\in X} f(x)$, we say that an optimizer exists if the infimum is attained, i.e. $\exists\, x^* \in X$ such that $J^* = f(x^*)$.

A *polyhedron* is the intersection of a finite set of open and/or closed halfspaces. A *polygon* is a union of finite number of polyhedra. We will adopt a similar notation to that presented in [11] with regards to the concept of *extended real valued* functions. Thus, a function f is allowed to take values in $\bar{\mathbb{R}} = [-\infty, \infty]$. Recall also that the *infimum* of a set S [12] is defined as the as the quantity m such that $\forall s \in S$, we have $m \leq s$ and $\forall \varepsilon > 0$ there exists $s \in S$ such that $s < m + \varepsilon$. Moreover, we introduce the notation $\inf_C f := \inf_{x \in C} f(x) := \inf \{ f(x) \mid x \in C \}$ and $\sup_C f := \sup_{x \in C} f(x) := \sup \{ f(x) \mid x \in C \}$. By convention we have $\inf_{\emptyset} f = \infty$ and $\sup_{\emptyset} f = -\infty$. We say that the infimum (supremum) of f over C is *attained* if $\arg\min_{x \in C} f(x) \neq \emptyset$ ($\arg\max_{x \in C} f(x) \neq \emptyset$). For a function $f : \mathbb{R}^n \to \bar{\mathbb{R}}$, the *domain* of f is defined as the set

$$\operatorname{dom}(f) := \{ x \in \mathbb{R}^n \mid -\infty < f(x) < \infty \}.$$

Whenever we refer to a function f or mapping F having a certain property, we implicitly mean that the property holds only on the domain of f or F, e.g. if we say that f is continuous, it is continuous at every $x \in \operatorname{dom}(f)$.

A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is *piecewise affine* (PWA) on its domain if $\operatorname{dom}(f)$ is the union of finitely many polyhedra, relative to each of which $f(\cdot)$ is affine. We say that the set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is *piecewise polyhedral* if the graph of F, defined as $\operatorname{gph}(F) := \{ (x, u) \mid u \in F(x) \}$, is a polygon. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is a *selection* of the set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ if $f(x) \in F(x)$ for all $x \in \operatorname{dom}(F)$. If $F : X \rightrightarrows Y$ is a mapping, then the restriction of F to the domain D is written $F|_D : D \rightrightarrows Y$.

Throughout we will use the superscript * to distinguish between optimizers and decision variables, e.g. for the problem $\min_x f(x)$, x is the decision variable and x^* denotes an optimizer.

Given the optimization problem

$$J^* := \inf_{u \in U} f(u),$$

we denote by $\varepsilon\text{-}\arg\min_{u \in U} f(u)$ the set of values of $u \in U$ for which $f(u) \leq J^* + \varepsilon$, that is,

$$\varepsilon\text{-}\arg\min_{u \in U} f(u) := \{ u \in U \mid f(u) \leq J^* + \varepsilon \}.$$

The following observation follows directly from the definition of the infimum of a set:

Lemma 1: Assume that $J^* > -\infty$ and that U is a non-empty set. Then $\varepsilon\text{-}\arg\min_{u \in U} f(u) \neq \emptyset, \forall \varepsilon > 0$.

III. ε-Optimal Solutions to Parametric Linear Programs with Strict and Non-strict Inequality Constraints

In this section we consider the problem of finding ε-optimal solutions to parametric linear programs with strict and non-strict inequalities. We propose a procedure that will be repeatedly used in subsequent sections for the purpose of obtaining sub-optimal solutions to optimal control problems with piecewise affine cost and polygonic constraints.

Consider problem

$$\mathbb{P}(x): \; J^*(x) := \inf_u \left\{ c^T u \mid (x, u) \in \mathcal{Z} \right\}, \tag{1a}$$

$$\mathcal{Z} := \left\{ (x, u) \in \mathbb{R}^n \times \mathbb{R}^m \;\middle|\; \begin{array}{ccc} Au + Bx & \leq & d \\ Eu + Fx & < & g \end{array} \right\}, \tag{1b}$$

which is to be solved for all $x \in \mathcal{X}$, where $\mathcal{X}$ is some polyhedral set, and c, A, B, d, E, F and g are matrices with suitable dimensions.

The constraint set $\mathcal{Z}$ defines the set valued map $U : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ given by $U(x) := \{ u \mid (x, u) \in \mathcal{Z} \}$ and hence, (1) can be written as

$$J^*(x) := \inf_{u \in U(x)} c^T u.$$

Assumption 1: We make the following standing assumption: For any parametric optimization problem that can be expressed as $z^*(\theta) := \inf_y \{ f(\theta, y) \mid (\theta, y) \in \mathcal{Y} \}, \forall \theta \in \Theta$, the sets $\mathcal{Y}$ and $\{ \theta \in \Theta \mid z^*(\theta) > -\infty \}$ are non-empty.

Returning to our original problem $\mathbb{P}(\cdot)$, the above assumption implies that $\mathcal{Z} \neq \emptyset$ and that there exists some polyhedral set of parameters on which J^* is lower bounded. Consequently, we re-define the set $\mathcal{X}$ as:

$$\mathcal{X} := \operatorname{Proj}_x \mathcal{Z} \cap \{ x \mid J^*(x) > -\infty \} = \operatorname{dom}(J^*). \tag{2}$$

For pLPs with only non-strict inequalities the following is well-known [13]–[16]:

Theorem 1 (Solution properties for pLPs): Consider the pLP

$$\hat{J}^*(x) := \min_u \left\{ c^T u \mid (x, u) \in \operatorname{cl}(\mathcal{Z}) \right\}, \tag{3}$$

which is to be solved for all values of $x \in \widehat{\mathcal{X}}$, where

$$\widehat{\mathcal{X}} := \operatorname{Proj}_x (\operatorname{cl}(\mathcal{Z})) \cap \left\{ x \;\middle|\; \hat{J}^*(x) > -\infty \right\} = \operatorname{dom}\left(\hat{J}^* \right).$$

$i)$ There exists a continuous and PWA function $u^* : \widehat{\mathcal{X}} \to \mathbb{R}^m$ that satisfies

$$u^*(x) \in \arg\min_u \left\{ c^T u \mid (x, u) \in \operatorname{cl}(\mathcal{Z}) \right\}.$$

$ii)$ The value function $\hat{J}^* : \widehat{\mathcal{X}} \to \mathbb{R}$ is continuous, convex, and piecewise affine.

Recall a fundamental result for support functions to convex sets, formulated as a lemma for clarity of presentation [17, page 112]:

Lemma 2: If $S \subset \mathbb{R}^m$ is a convex set and given $c \in \mathbb{R}^m$, then

$$\inf_{u \in S} c^T u = \inf_{u \in \operatorname{cl}(S)} c^T u = \inf_{u \in \operatorname{relint}(S)} c^T u.$$

From the above lemma it is clear that $J^*(x) = \hat{J}^*(x)$ for all $x \in \mathcal{X} = \operatorname{dom}(J^*)$.

A parametric optimization is said to be a *piecewise pLP* if the set of parameters for which the infimum is bounded is represented by a union of a finite number of polyhedra, relative to each of which the problem reduces to a pLP. Consider $\mathbb{P}(\cdot)$ and define the piecewise pLP

$$\mathbb{P}_\varepsilon(x): \; V_\varepsilon^*(x) := \min_{(u,t)} \{ t \mid (x, u, t) \in \mathcal{Z}_\varepsilon \}, \tag{4a}$$

$$\mathcal{Z}_\varepsilon := \left\{ (x, u, t) \;\middle|\; \begin{array}{ccc} Au + Bx & \leq & d \\ Eu + Fx & \leq & g + \mathbf{1}t \\ c^T u & \leq & \hat{J}^*(x) + \varepsilon \end{array} \right\}, \tag{4b}$$

where $\mathbf{1}$ is a vector of ones, which is to be solved for all values of $x \in \mathcal{X}_\varepsilon$, where

$$\mathcal{X}_\varepsilon := \operatorname{Proj}_x \mathcal{Z}_\varepsilon \cap \widehat{\mathcal{X}}. \tag{5}$$

We propose the following theorem for the purpose of obtaining ε-optimal solutions to pLPs with strict and non-strict inequality constraints:

Theorem 2: Consider the optimization problems given in (1) and (4). The following holds:

i) $\mathcal{X}_\varepsilon = \widehat{\mathcal{X}} = \mathrm{dom}\left(\hat{J}^*\right) \supseteq \mathcal{X} = \mathrm{dom}\,(J^*)$.

ii) Given any $\varepsilon > 0$ we have that (4) attains its minimum $\forall x \in \mathcal{X}_\varepsilon$, and that given any $x \in \mathcal{X}$

$$\forall (u^*(x), t^*(x)) \in \arg\min_{(u,t)} \{ t \mid (x,u,t) \in \mathcal{Z}_\varepsilon \} \Rightarrow$$
$$u^*(x) \in \varepsilon\text{-}\arg\min_{u \in U(x)} c^T u$$

iii) The function $V_\varepsilon^*(\cdot)$ is continuous and piecewise affine on $\mathcal{X}_\varepsilon$.

iv) There exists a minimizer function $u^* : \mathcal{X}_\varepsilon \to \mathbb{R}^n$ that is continuous and PWA.

Proof:

i) That $\widehat{\mathcal{X}} \supseteq \mathcal{X}$ follows from $\mathrm{Proj}_x\,(\mathrm{cl}\,(\mathcal{Z})) \supseteq \mathrm{Proj}_x\,\mathcal{Z}$ and $\left\{ x \mid \hat{J}^*(x) > -\infty \right\} = \{ x \mid J^*(x) > -\infty \}$ (recall that $\hat{J}^*(x) = \infty$ if $x \notin \widehat{\mathcal{X}}$), hence,

$$\widehat{\mathcal{X}} = \mathrm{Proj}_x\,(\mathrm{cl}\,(\mathcal{Z})) \cap \left\{ x \mid \hat{J}^*(x) > -\infty \right\} \supseteq$$
$$\mathrm{Proj}_x\,\mathcal{Z} \cap \{ x \mid J^*(x) > -\infty \} = \mathcal{X}.$$

That $\mathcal{X}_\varepsilon = \widehat{\mathcal{X}}$ holds trivially by noting that fixing $t = 0$ and $\varepsilon = 0$ for all $x \in \widehat{\mathcal{X}}$ renders the minimizer $u^*(\cdot)$ of (3) feasible also for (4).

ii) Given any $x \in \mathcal{X}_\varepsilon$, (4) is an LP and consequently attains its minimum if the infimum is bounded, which is true on $\mathcal{X}_\varepsilon$ by construction. Lemma 1 and Assumption 1 ensures that there always exists ε-optimal solutions, i.e.

$$\forall x \in \mathcal{X}\ \exists \tilde{u} \in \varepsilon\text{-}\arg\min_{u \in U(x)} c^T u.$$

Thus, for any $x \in \mathcal{X}$ and for all $\tilde{u} \in \varepsilon\text{-}\arg\min_{u \in U(x)} c^T u$ there exists some $\gamma(\tilde{u}) < 0 \in \mathbb{R}$ such that

$$\emptyset \neq \left\{ u \;\middle|\; \begin{array}{rcl} Au + Bx & \leq & d \\ Eu + Fx & \leq & g + \mathbf{1}\gamma(\tilde{u}) \\ c^T u & \leq & \hat{J}^*(x) + \varepsilon \end{array} \right\} \subseteq U(x),$$

which immediately implies $t^*(x) < 0$ for all $x \in \mathcal{X}$, and consequently $u^*(x) \in \varepsilon\text{-}\arg\min_{u \in U(x)} c^T u$ for all $x \in \mathcal{X}$.

iii) Define a new parameter y and write (4) as

$$\bar{V}_\varepsilon^*(x,y) := \min_{(\bar{u},\bar{t})} \left\{ \bar{t} \mid (x,y,\bar{u},\bar{t}) \in \bar{\mathcal{Z}}_\varepsilon \right\},$$

$$\bar{\mathcal{Z}}_\varepsilon := \left\{ (x,y,\bar{u},\bar{t}) \;\middle|\; \begin{array}{rcl} A\bar{u} + Bx & \leq & d \\ E\bar{u} + Fx & \leq & g + \mathbf{1}\bar{t} \\ c^T \bar{u} & \leq & y + \varepsilon \end{array} \right\}.$$

Clearly, the above problem is a pLP, and consequently, $\bar{V}_\varepsilon^*(\cdot)$ is continuous and piecewise affine. Letting $y = \hat{J}^*(\cdot)$ we see that $\bar{V}_\varepsilon^*(\cdot, \hat{J}^*(\cdot))$ is a composition of continuous functions and therefore also a continuous function. Moreover, composition of PWA functions is a PWA function. We clearly also have $V_\varepsilon^*(x) = \bar{V}_\varepsilon^*(x, \hat{J}^*(x))$ for all $x \in \mathcal{X}_\varepsilon$.

iv) Following the same argument as in *iii)* we have that there exists an optimal selection $\bar{u}^*(\cdot)$ that is continuous and PWA, and hence the same holds for $u^*(\cdot) = \bar{u}^*(\cdot, \hat{J}^*(\cdot))$.

■

Theorem 2 provides a procedure for obtaining ε-optimal solutions to pLPs with strict and non-strict inequality constraints. The pLP is first solved over the closure of $\mathcal{Z}$ to obtain the function $\hat{J}^*(\cdot)$. Then solving (4) ensures that we obtain a function $u^*(\cdot)$ such that $c^T u^*(x) \leq \hat{J}^*(x) + \varepsilon$ for all $x \in \mathcal{X}$.

One important detail that should be emphasized is that we can restrict the domain of the selection $u^*(\cdot)$ to $\mathcal{X}$, which is possible since $\mathcal{X}_\varepsilon \supseteq \mathcal{X}$. Thus, we let $u^* : \mathcal{X}_\varepsilon \to \mathbb{R}^m$ be redefined to $u^* : \mathcal{X} \to \mathbb{R}^m$. The restriction of the domain is important because in subsequent sections we want to apply this procedure to the minimization of discontinuous piecewise affine functions, thus slightly enlarging the domain may introduce arbitrary large errors if we try to select the minimum of several affine functions. In the sequel, $(u_\varepsilon^*(\cdot), t_\varepsilon^*(\cdot))$ will denote a continuous and optimal selection for (4), whose domain is restricted to $\mathcal{X}$.

IV. ε-Optimal solutions for PWA functions

In the previous section we proposed a procedure for obtaining ε-optimal solutions to pLPs with strict and non-strict inequalities. In this section the procedure is repeatedly applied for the purpose of finding ε-optimal solutions to minimization of PWA functions over polygonic sets.

Consider the problem of minimizing $f(x,\cdot)$, where $f(\cdot)$ is piecewise affine. We will represent f in the following manner:

$$f(x,u) = f_i(x,u) \quad \text{if} \quad (x,u) \in P_i \subset \mathbb{R}^n \times \mathbb{R}^m,$$

where $i \in \{1,2,\ldots,I\}$, each f_i is affine and each P_i is a polyhedron, thus the domain of f is the polygon $\mathcal{P} := \mathrm{dom}\,(f) = \cup_{i=1}^I P_i$. Note that this implies that for each pair $(i,j) \in \{1,2,\ldots,I\} \times \{1,2,\ldots,I\}$ we have $f_i(x,u) = f_j(x,u)$, $\forall (x,u) \in P_j \cap P_i$.

Consider the following optimization problem:

$$J^*(x) := \inf_u \{ f(x,u) \mid (x,u) \in \mathcal{P} \}. \tag{6}$$

We can clearly represent (6) as

$$J^*(x) := \min_{i \in \{1,2\ldots,I\}} \left\{ \inf_u \{ f_i(x,u) \mid (x,u) \in P_i \} \right\}. \tag{7}$$

Observing that $J_i^*(x) := \inf_u \{ f_i(x,u) \mid (x,u) \in P_i \}$ is a pLP with strict and non-strict inequalities for each $i \in \{1,2,\ldots,I\}$ we let $\left\{ (u_{i,\varepsilon}^*(\cdot), t_{i,\varepsilon}^*(\cdot)) \mid i \in \{1,2,\ldots,I\} \right\}$ denote a set of continuous selections where each pair $(u_{i,\varepsilon}^*(\cdot), t_{i,\varepsilon}^*(\cdot))$ optimizes the corresponding piecewise pLP defined by (4). Recall also from the previous section that the domain of each $u_{i,\varepsilon}^*(\cdot)$ is restricted to the domain of $J_i^*(\cdot)$, and hence, $\mathrm{dom}\left(f_i(\cdot, u_{i,\varepsilon}^*(\cdot))\right) = \mathrm{dom}\,(J_i^*)$.

Theorem 3: Consider the optimization problem given in (6).

i) For any $\varepsilon > 0$, $x \in \mathrm{dom}\,(J^*)$ and the problem

$$J_\varepsilon^*(x) := \min_{i \in \{1,\ldots,I\}} \{ f_i(x, u_{i,\varepsilon}^*(x)) \}, \tag{8}$$

we have that

$$j \in \arg\min_{i\in\{1,\ldots,I\}} \{f_i(x,u^*_{i,\varepsilon}(x))\} \Rightarrow$$
$$u^*_{j,\varepsilon}(x) \in \varepsilon\text{-}\arg\min_u \{f(x,u) \mid (x,u)\in\mathcal{P}\}.$$

$ii)$ $\operatorname{dom}(J^*_\varepsilon) = \operatorname{dom}(J^*)$.

Proof:

$i)$ Since $\forall x \in \operatorname{dom}(J^*_i)$ and $\forall i \in \{1,2,\ldots,I\}$ we have
$$u^*_{i,\varepsilon}(x) \in \varepsilon\text{-}\arg\min_u \{f_i(x,u) \mid (x,u)\in P_i\},$$
the assertion trivially holds.

$ii)$ This follows by construction; the domain for each $u^*_{i,\varepsilon}(\cdot)$ is restricted to the domain of J^*_i, hence the domain of J^*_ε is equal to $\cup_{i=1}^I \operatorname{dom}(J^*_i)$, which is precisely the domain of J^*. ■

V. ε-OPTIMAL SOLUTIONS TO INF-SUP OF PWA FUNCTIONS

It is apparent from the two preceding sections that a pLP with strict and non-strict inequalities can be viewed as a sub-problem of minimizing a PWA function over polygonic constraints. In this section we extend the approach to $\inf\text{-}\sup$ problems and now minimization of PWA functions become our sub-problems.

Consider the problem
$$J^*(x) := \inf_{u\in\mathcal{U}(x)} \sup_{w\in\mathcal{W}(x,u)} f(x,u,w) \tag{9}$$
where again we consider the problem where $f(\cdot)$ is PWA and defined on the polygon
$$\mathcal{P} := \cup_{i=1}^I P_i,\ P_i \subset \mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^p, \forall i\in\{1,2\ldots,I\}.$$
and where the set $\mathcal{U}(\cdot)$ is defined from
$$\mathcal{Z} := \{(x,u) \mid (x,u)\in\mathcal{Y}, (x,u,w)\in\Pi\ \forall w\in\mathcal{W}(x,u)\},$$
$$\mathcal{U}(x) := \{u \mid (x,u)\in\mathcal{Z}\},$$
where $\mathcal{Y}$, Π, and $\operatorname{gph}(\mathcal{W})$ are non-empty polygons, and $\mathcal{W}(x,u)\neq\emptyset$ for all $(x,u)\in\mathcal{Y}$. The reader is referred to [18] for details on how to compute $\mathcal{Z}$. Define also the sets
$$\mathcal{X} := \operatorname{Proj}_x \mathcal{Z} = \{x \mid \exists u : (x,u)\in\mathcal{Z}\},$$
$$\Gamma := \{(x,u,w) \mid w\in\mathcal{W}(x,u)\} = \operatorname{gph}(\mathcal{W}) =: \cup_{j=1}^T \Gamma_j.$$

The problem (9) can be divided into a supremum and an infimum problem as
$$V^*(x,u) := \sup_w \{f(x,u,w) \mid (x,u,w)\in\Gamma\}, \forall (x,u)\in\mathcal{Z},$$
$$J^*(x) = \inf_u \{V^*(x,u) \mid (x,u)\in\mathcal{Z}\}, \forall x\in\mathcal{X}.$$

In this section we view problem (9) from a game theoretic point in the sense that we choose u and our adversary chooses w. We are therefore not concerned with attaining a maximizing w, but only a minimizing u. Define the polygon
$$\mathcal{F} := \cup_{h=1}^H F_h$$
where the polyhedra $\{F_1, F_2, \ldots, F_H\}$ covers the set $\Gamma\cap\mathcal{P}$, and each polyhedron F_h is a member of the set
$$\left\{ \Gamma_j \cap P_i \;\middle|\; \begin{array}{c} (i,j)\in\{1,2,\ldots,I\}\times\{1,2,\ldots,T\}, \\ \Gamma_j\cap P_i \neq \emptyset \end{array} \right\}.$$

Hence, we can restrict our PWA function f to the domain for which Assumption 1 is valid by defining:
$$f(x,u.w) = z_h(x,u,w) \quad \text{if} \quad (x,u,w)\in F_h,$$
where $z_h(x,u,w) := f_i(x,u,w)$ if $(x,u,w)\in P_i$.

For each $h\in\{1,2,\ldots,H\}$ define the pLPs:
$$\hat{V}^*_h(x,u) := \max_w \{z_h(x,u,w) \mid (x,u,w)\in\operatorname{cl}(F_h)\}, \tag{11a}$$
$$V^*_h(x,u) := \sup_w \{z_h(x,u,w) \mid (x,u,w)\in F_h\}. \tag{11b}$$

Theorem 4: The following holds for all $h \in \{1,2\ldots,H\}$:
$$V^*_h(x,u) = \hat{V}^*_h(x,u), \quad \forall (x,u)\in\operatorname{dom}(V^*_h).$$
Moreover,
$$V^*(x,u) = \max_{h\in\{1,2,\ldots,H\}} \{\hat{V}^*_h|_{\operatorname{dom}(V^*_h)}(x,u)\}.$$

Proof: The first assertion holds trivially from the fact that (11a) is a pLP and from the equality of the supremum and maximum over respectively F_h and $\operatorname{cl}(F_h)$, cf. Lemma 2. Having the first statement established automatically ensures that the second assertion is correct, since, for each $h\in\{1,2,\ldots,H\}$, we restrict $\hat{V}^*_h(\cdot)$ to the domain of $V^*_h(\cdot)$. ■

Since we now have an exact expression for $V^*(\cdot)$, we can now apply the procedure from the previous section for the purpose of obtaining an ε-optimal solution to our problem. Recalling that $V^*(\cdot)$ is PWA and defined on a polygon $\mathcal{R} = \cup_{k=1}^K R_k$, that is,
$$V^*(x,u) = V_k(x,u) \quad \text{if} \quad (x,u)\in R_k,$$
then $J^*_\varepsilon(\cdot)$ is defined as
$$J^*_\varepsilon(x) := \min_{k\in\{1,2,\ldots,K\}} \{V_k(x,u^*_{k,\varepsilon}(x))\}. \tag{12}$$

Theorem 5: We have that $J^*_\varepsilon(x) \le J^*(x)+\varepsilon$, $\forall x \in \operatorname{dom}(J^*)$, and
$$\varepsilon\text{-}\arg\min_u \{V^*(x,u) \mid (x,u)\in\mathcal{Z}\} \neq \emptyset, \quad \forall x\in\operatorname{dom}(J^*).$$

Proof: Both statements are confirmed by construction and consequences of Theorem 2, 3, and 4. ■

VI. ε-OPTIMAL SOLUTIONS TO INF-SUP OPTIMAL CONTROL OF PWA SYSTEMS

In this section we use the results of the previous sections to obtain ε-optimal solutions to robust optimal control problems for discontinuous PWA systems subject to state- and input-dependent disturbances. We recall the problem setup from [2].

A. Problem setup

Consider the discrete-time system of the form:
$$x^+ = g(x,u,w),$$
where x is the state, x^+ is the successor state, u is the input, $g(\cdot)$ is assumed piecewise affine on the polygon $\mathcal{P}$, $w\in\mathcal{W}(x,u)\subset\mathbb{R}^p$ is a time varying disturbance. The state and input are subject to constraints $(x,u)\in\mathcal{Y}\subset\mathbb{R}^n\times\mathbb{R}^m$, where we assume that $\mathcal{Y}$ is a polygon. The constraints define the set-valued map $\mathcal{U}(x) := \{u \mid (x,u)\in\mathcal{Y}\}$.

Let $\pi := \{\mu_0(\cdot), \mu_1(\cdot), \ldots, \mu_{N-1}(\cdot)\}$ denote a control policy (i.e. $\mu_i : \mathbb{R}^n \leftarrow \mathbb{R}^m$) over the horizon N and let $\mathbf{w} := \{w_0, w_1, \ldots, w_{N-1}\}$ denote a sequence of disturbances. Moreover, let $\phi(i; x, \pi, \mathbf{w})$ denote the solution to $x^+ = g(x, u, w)$ at time-step i for the initial state x, control policy π and disturbance sequence $\mathbf{w}$.

The cost is defined as

$$J_N(x, \pi, \mathbf{w}) := J_f(x_N) + \sum_{i=0}^{N-1} l(x_i, u_i)$$

where $x_i := \phi(i; x, \pi, \mathbf{w})$ and $u_i := \mu_i(x_i)$, $\forall i$. The stage cost $l(\cdot)$ and terminal cost $J_f(\cdot)$ are assumed to be PWA ($p \in \{1, \infty\}$):

$$\begin{aligned} l(x, u) &: = \|Qx\|_p + \|Ru\|_p, \\ J_f(x) &: = \|Px\|_p, \end{aligned}$$

where P, Q, and R are suitably defined weight matrices.

The optimal control problem considered is given by

$$\mathbb{P}_N(x): \ J_N^*(x) := \inf_{\pi \in \Pi_N(x)} \sup_{\mathbf{w} \in \mathbf{W}(x,\pi)} J_N(x, \pi, \mathbf{w}), \quad (13)$$

where the set of admissible disturbance sequences is given by

$$\mathbf{W}(x, \pi) := \{\mathbf{w} \mid w_i \in \mathcal{W}(x_i, u_i), \ i = 0, 1, \ldots, N-1\},$$

and the set of admissible control polices is

$$\Pi_N(x) := \left\{ \pi \ \middle| \ \begin{array}{c} (x_i, u_i) \in \mathcal{Y}, \ i = 0, 1, \ldots, N-1, \\ x_N \in X_f, \ \forall \mathbf{w} \in \mathbf{W}(x, \pi) \end{array} \right\}.$$

In the sequel, we denote by X_N the set of initial states for which there exists an admissible control policy, i.e. $X_N := \{x \mid \Pi_N(x) \neq \emptyset\}$. In addition we make the following assumptions in order to ensure that $\mathbb{P}_N(x)$ is well defined for all $x \in X_N$:

A1: The system $g : \mathcal{P} \to \mathbb{R}^n$ is PWA on the polygon $\mathcal{P}$.
A2: The sets $\mathcal{Y}$ and X_f are non-empty polygons.
A3: For all $(x, u) \in \mathcal{Y}$, the set $\mathcal{W}(x, u)$ is non-empty.
A4: $\mathrm{gph}\,(\mathcal{W})$ is a non-empty polygon.
A5: $J_N^*(x)$ is bounded $\forall x \in X_N$.

Thus, in comparison to [2] several assumptions are relaxed (note that we use the definition in [11] for continuity of a set valued map):

- $i)$ We do not assume that $g(\cdot)$ is continuous.
- $ii)$ $\mathcal{Y}$ and X_f are not required to have the origin in the interior.
- $iii)$ The set-valued map $x \mapsto \mathcal{U}(x)$ is not required to be continuous and bounded on bounded sets.
- $iv)$ The set-valued map $(x, u) \mapsto \mathcal{W}(x, u)$ is not required to be continuous.
- $v)$ The solution to $\mathbb{P}_N(x)$ is not assumed to exist $\forall x \in X_N$.

Remark 1: It should be noted that in [2] the majority of the assumptions above are made for the purpose of being able to directly apply the topological results in [18].

B. Sub-optimal solution via dynamic programming.

Dynamic programming [2] provides a recursive procedure for computing sequentially the partial return functions $J_j^*(\cdot)$ (defined in (13) with $N = j$), the associated set-valued control laws $\kappa_j(\cdot)$ as well as their domains (here j denotes 'time-to-go' so that $\kappa_j(\cdot) = \mu_{N-j}^*(\cdot)$ if $j \in \{1, \ldots, N-1\}$ and $\kappa_N(\cdot) = \mu_0^*(\cdot)$). The domain of $J_j^*(\cdot)$ and $\kappa_j(\cdot)$ is X_j, the set of states that can be robustly steered to the terminal set X_f in j steps or less. Define also

$$g(x, u, \mathcal{W}(x, u)) := \{g(x, u, w) \mid w \in \mathcal{W}(x, u)\}.$$

The solution to $\mathbb{P}_N(x)$ may be obtained as follows. For all $j \in \{1, 2, \ldots\}$, j denotes "time-to-go", and the partial return function $J_j^*(\cdot)$, the control law $\kappa_j(\cdot)$, and the controllability set X_j are given by:

$$J_j^*(x) = \inf_{u \in \mathcal{U}(x)} \sup_{w \in \mathcal{W}(x,u)} \bar{J}_j(x, u, w), \ \forall x \in X_j, \quad (14a)$$

$$\bar{J}_j(x, u, w) := \left\{ \begin{array}{c} \ell(x, u) + \\ J_{j-1}^*(x^+) \end{array} \middle| g(x, u, \mathcal{W}(x, u)) \subseteq X_{j-1} \right\},$$

$$\kappa_j(x) = \arg\min_{u \in \mathcal{U}(x)} \sup_{w \in \mathcal{W}(x,u)} \bar{J}_j(x, u, w), \quad (14b)$$

$$X_j = \{x \mid \exists u \in \mathcal{U}(x) \text{ s.t. } g(x, u, \mathcal{W}(x, u)) \subseteq X_{j-1}\}, \quad (14c)$$

with boundary conditions

$$J_0^*(x) = J_f(x), \qquad X_0 = X_f. \quad (14d)$$

The conditions $g(x, u, \mathcal{W}(x, u)) \subseteq X_{j-1}$ *and* $u \in \mathcal{U}(x)$ in (14) may be expressed as

$$(x, u) \in \Sigma_j := \left\{ (x, u) \in \mathcal{Y} \ \middle| \ \begin{array}{c} g(x, u, w) \in X_{j-1} \\ \forall w \in \mathcal{W}(x, u) \end{array} \right\},$$

in which case X_j can be interpreted as the projection of the set Σ_j onto X, i.e.

$$X_j = \{x \mid \exists u : (x, u) \in \Sigma_j\}.$$

The reader is referred to [18] for details on how to compute Σ_j. In order to analyze $\mathbb{P}_N(x)$, it is convenient to introduce the functions $V_j^*(\cdot)$, $j = 1, 2, \ldots, N-1$, defined by

$$V_j^*(x, u) := \sup_{w \in \mathcal{W}(x,u)} J_j^*(g(x, u, w)).$$

Note that we are interested in values of the functions $V_j^*(\cdot)$, $j = 1, 2, \ldots, N-1$, and the sets Σ_j, $j = 1, 2, \ldots, N-1$. The recursion (14a)-(14c) may therefore be rewritten as

$$V_{j-1}^*(x, u) = \sup_w \{J_{j-1}^*(g(x, u, w)) \mid w \in \mathcal{W}(x, u)\},$$
$$\text{for all} \quad (x, u) \in \Sigma_j, \quad (15a)$$

$$J_j^*(x) = \inf_u \{\ell(x, u) + V_{j-1}^*(x, u) \mid (x, u) \in \Sigma_j\},$$
$$\text{for all} \quad x \in X_j, \quad (15b)$$

$$\kappa_j(x) = \arg\min_u \{\ell(x, u) + V_{j-1}^*(x, u) \mid (x, u) \in \Sigma_j\},$$
$$\text{for all} \quad x \in X_j, \quad (15c)$$

$$\Sigma_j = \{(x, u) \in \mathcal{Y} \mid g(x, u, \mathcal{W}(x, u)) \subseteq X_{j-1}\}, \quad (15d)$$

$$X_j = \{x \mid \exists u : (x, u) \in \Sigma_j\}. \quad (15e)$$

For each j we propose to use the approximate procedure presented in the previous sections (Theorems 2–4) when solving (15b) in order to ensure that the set $\kappa_j(\cdot)$ (15c) is a non-empty set. Thus, for each j for which $\kappa_j(x) = \emptyset$ for some $x \in X_j$, we compute the approximation $\kappa_{j,\varepsilon}(\cdot)$, that is, $\kappa_{j,\varepsilon}(\cdot)$ is a *selection* from the set-valued map $x \mapsto \varepsilon\text{-}\arg\min_u \left\{\ell(x,u) + V_{j-1}^*(x,u) \mid (x,u) \in \Sigma_j\right\} \subseteq \mathbb{R}^m$.

Two approaches are natural when considering the dynamic programming recursion; the first is the one outlined above, namely using the exact expressions for the functions $\{V_j^*(\cdot)\}_{j=0}^{N-1}$ and $\{J_j^*(\cdot)\}_{j=0}^{N}$ and use Theorems 2–4 to compute $\{\kappa_{j,\varepsilon}(\cdot)\}_{j=1}^{N}$. The second approach is to use the approximate value-function in the dynamic programming recursion:

$$\begin{aligned} V_{j-1,\varepsilon}^*(x,u) &= \sup_w \left\{ J_{j-1,\varepsilon}^*(g(x,u,w)) \mid w \in \mathcal{W}(x,u) \right\}, \\ &\qquad\qquad \text{for all} \quad (x,u) \in \Sigma_j, \\ \kappa_{j,\varepsilon}(x) &\in \varepsilon\text{-}\arg\min_u \left\{ \ell(x,u) + V_{j-1,\varepsilon}^*(x,u) | (x,u) \in \Sigma_j \right\}, \\ &\qquad\qquad \text{for all} \quad x \in X_j, \\ J_{j,\varepsilon}^*(x) &= l(x,\kappa_{j,\varepsilon}(x)) + V_{j-1,\varepsilon}^*(x,\kappa_{j,\varepsilon}(x)), \quad \forall x \in X_j. \end{aligned}$$

With this approach an error bound is easily derived, as demonstrated by the following theorem:

Theorem 6: $J_{N,\varepsilon}^*(x) \leq J_N^*(x) + N\varepsilon$.

Proof: We show this by induction. We verify the induction base by carrying out the first iteration of the dynamic programming recursion:

$$\begin{aligned} V_0^*(x,u) &= \sup_w \left\{ J_0^*(g(x,u,w)) \mid w \in \mathcal{W}(x,u) \right\}, \\ &\qquad\qquad \text{for all} \quad (x,u) \in \Sigma_1, \\ \kappa_{1,\varepsilon}(x) &\in \varepsilon\text{-}\arg\min_u \left\{ \ell(x,u) + V_0^*(x,u) \mid (x,u) \in \Sigma_1 \right\}, \\ &\qquad\qquad \text{for all} \quad x \in X_1, \\ J_{1,\varepsilon}^*(x) &= l(x,\kappa_{1,\varepsilon}(x)) + V_0^*(x,\kappa_{1,\varepsilon}(x)) \leq J_1^*(x) + \varepsilon. \end{aligned}$$

Assuming that the bound holds for $N = j$, i.e.

$$J_{j,\varepsilon}^*(x) \leq J_j^*(x) + j\varepsilon,$$

we must verify that the bound also holds for $N = j+1$. We get:

$$\begin{aligned} V_{j,\varepsilon}^*(x,u) &= \sup_w \left\{ J_{j,\varepsilon}^*(g(x,u,w)) \mid w \in \mathcal{W}(x,u) \right\} \\ &\leq \sup_w \left\{ J_j^*(g(x,u,w)) \mid w \in \mathcal{W}(x,u) \right\} + j\varepsilon \\ &= V_j^*(x,u) + j\varepsilon, \quad \forall (x,u) \in \Sigma_{j+1}, \end{aligned}$$

and

$$\begin{aligned} &\kappa_{j+1,\varepsilon}(x) \in \\ &\quad \varepsilon\text{-}\arg\min_u \left\{ l(x,u) + V_{j,\varepsilon}^*(x,u) \mid (x,u) \in \Sigma_{j+1} \right\}, \end{aligned}$$

and since

$$\begin{aligned} &\inf_u \left\{ l(x,u) + J_{j,\varepsilon}^*(x,u) \mid (x,u) \in \Sigma_{j+1} \right\} \leq \\ &\overbrace{\inf_u \left\{ l(x,u) + V_j^*(x,u) \mid (x,u) \in \Sigma_{j+1} \right\}}^{J_{j+1}^*(x)} + j\varepsilon \end{aligned}$$

we get

$$\begin{aligned} J_{j+1,\varepsilon}^*(x) &= l(x,\kappa_{j+1,\varepsilon}(x)) + V_{j,\varepsilon}^*(x,\kappa_{j+1,\varepsilon}(x)) \\ &\leq J_{j+1}^*(x) + (j+1)\varepsilon. \end{aligned}$$

■

With this approach it is clear that the degree of sub-optimality can be a priori specified by choosing ε.

VII. Conclusion

A method for obtaining approximate solutions to robust optimal control of discontinuous PWA systems has been presented. This was achieved by repeatedly applying a procedure, that obtained ε-optimal solutions to pLPs with strict and non-strict inequality constraints, in a dynamic programming approach. It has been demonstrated that ε-optimal solutions always exists, a bound on the total error for the approximate dynamic programming has been given and the degree of sub-optimality can be a priori specified. Future research includes stability considerations for the proposed control scheme.

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References

[1] F. Borrelli, M. Baotić, A. Bemporad, and M. Morari, “Dynamic programming for constrained optimal control of discrete-time linear hybrid systems,” *Automatica*, vol. 41, pp. 1709–1721, 2005.

[2] S. V. Raković, E. C. Kerrigan, and D. Q. Mayne, “Optimal control of constrained piecewise affine systems with state- and input-dependent disturbances,” in *Proceedings of the 16th International Symposium on Mathematical Theory of Networks and Systems*, Katholieke Universiteit Leuven, Belgium, 2004.

[3] D. Q. Mayne and S. V. Raković, “Optimal control of constrained piecewise affine systems using reverse transformation,” in *Proceedings of the 41st IEEE Conference on Decision and Control*, vol. 2, Las Vegas, USA, 2002, pp. 1546–1551.

[4] A. Bemporad, F. Borrelli, and M. Morari, “Piecewise linear optimal controllers for hybrid systems,” in *Proc. American Contr. Conf.*, Chicago, IL, 2000, pp. 1190–1194.

[5] ——, “Optimal Controllers for Hybrid Systems: Stability and Piecewise Linear Explicit Form,” in *IEEE Conference on Decision and Control*, Sydney, Australia, 2000.

[6] F. Borrelli, “Discrete time constrained optimal control,” Ph.D. dissertation, Swiss Federal Institute of Technology, 2002.

[7] D. M. de la Peña, T. Alamo, A. Bemporad, and E. F. Camacho, “A dynamic programming approach for determining the explicit solution of mpc controllers,” in *Proceedings of the 43rd IEEE Conference on Decision and Control*, Paradise Island, Bahamas, 2002.

[8] M. Diehl and J. Björnberg, “Robust dynamic programming for min-max model predictive control of constrained uncertain systems,” *IEEE Trans. Automatic Control*, vol. 49, no. 12, pp. 2253–2257, 2004.

[9] A. Bemporad and M. Morari, “Control of systems integrating logic, dynamics, and constraints,” *Automatica*, vol. 35, no. 3, pp. 407–427, 1999.

[10] R. Goebel and A. R. Teel, “Solutions to hybrid inclusions via set and graphical convergence with stability theory applications,” *Automatica*, vol. 42, pp. 573–587, 2006.

[11] R. T. Rockafellar and R. J.-B. Wets, *Variational Analysis*, ser. Grundlehren der mathematischen Wissenschaften. Berlin: Springer-Verlag, 1998, vol. 317.

[12] H. Jeffreys and B. S. Jeffreys, *Upper and Lower Bounds*, 3rd ed. Cambridge, England: Cambridge University Press, 1988.

[13] T. Gal and J. Nedoma, “Multiparametric linear programming,” *Management Science*, vol. 18, pp. 406–442, 1972.

[14] B. Bank, J. Guddat, D. Klatte, B. Kummer, and K. Tammer, *Non-linear Parametric Optimization*. Berlin: Birkhäuser, 1983.

[15] G. B. Dantzig, J. Folkman, and N. Z. Shapiro, “On the continuity of the minimum set of a continuous function,” *Journal of Mathematical Analysis and Applications*, vol. 17, no. 3, pp. 519–548, 1967.

[16] F. Borrelli, A. Bemporad, and M. Morari, “A geometric algorithm for multi-parametric linear programming,” *Journal of Optimization Theory and Applications*, vol. 118, no. 3, pp. 515–540, 2003.

[17] R. T. Rockafellar, *Convex Analysis*. Princeton, New Jersey: Princeton Univeristy Press, 1972.

[18] S. V. Raković, E. C. Kerrigan, D. Q. Mayne, and J. Lygeros, “Reachability analysis of discrete-time systems with disturbances,” *IEEE Transactions on Automatic Control*, vol. 51, pp. 546–561, 2006.