Constrained Optimal Control of Large-Scale Systems via Model Reduction

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Outline

- Motivation
- Large-Scale Systems
- Model predictive control (MPC)
  - Explicit solution via multi-parametric quadratic programming
- Projection based model reduction
  - Goal-oriented model-based optimization
- Numerical example
- Conclusions and further work
Motivation

• Constrained optimal control for systems that are
  – described by models of high order
  – characterized by fast dynamics
  – e.g. mechatronics, MEMS, rotating machinery, acoustics, flow control
Large-Scale Systems

• CFD models
  – High fidelity but expensive
    • 2D-Euler: ~ $10^4$ states
  – Unsuitable for:
    • (Optimal and model based) control design
    • Real-time applications

• Model reduction
  – From ~$10^4$ to ~$10^1$ states
MPC

+ Ability to handle multivariable systems
+ Ability to enforce *constraints* on variables
  ! Requires on line optimization
  ! Limited to processes with low sampling rates and/or slow dynamics

Allgöwer, 2004
Linear MPC

• Minimize quadratic cost at every sampling instant:

\[
\min_U x_{k+N}^T P x_{k+N} + \sum_{i=0}^{N-1} (x_{k+i}^T Q x_{k+i} + u_{k+i}^T R u_{k+i})
\]

\[
U = [ u_k \ u_{k+1} \ \ldots \ u_{k+N-1} ]^T
\]

subject to

– plant model

\[
x_{k+1} = Ax_k + Bu_k
\]

\[
y_k = Cx_k
\]

– constraints on input and outputs/states

\[
 u_{\min} \leq u \leq u_{\max}
\]

\[
y_{\min} \leq y \leq y_{\max}
\]
Explicit MPC
Bemporad et al., 2002, Tøndel et al., 2003

• Define $z \triangleq U + H^{-1} F^T x_k$

• Multiparametric program in $Z$, where $x$ and $U$ are parameters:

$$\min_z \left\{ \frac{1}{2} z^T H z \right\}$$

subject to: $Gz \leq W + S x_k$,

• $z^*$ remains optimal in a neighborhood of $x_k$

• Solve optimization problem ***off line*** for "all" $x$

• $U^*$ continuous piecewise affine function of current state:

$$u = K_1^i x + K_2^i$$
Explicit MPC: example

- State space partition
- \((K_1^i, k_2^i)\) constant within each polyhedron
Explicit MPC

- On line effort reduced to evaluating piecewise affine function of current state $x$
  - Determine critical region
+ On line computational time can be reduced to the microsecond-millisecond range
+ Low complexity, easily verifiable code
  ! Complexity of partition/controller increases rapidly with number of parameters/states
Projection Based Model Reduction

\[ \dot{x} = Ax + Bu \]
\[ y = Cx \]
\[ x \in \mathbb{R}^n \]

\[ \dot{x}_r = A_r x_r + B_r u \]
\[ y_r = C_r x_r \]
\[ x_r \in \mathbb{R}^r \]

\[ r \ll n \]

\[ A_r = V^T AV, \quad B_r = V^T B, \quad C_r = CV \]
Projection Basis

- Find reduced-order basis

\[ V = [V_1, \ldots, V_n] \in \mathbb{R}^{n \times r} \]

so that system dynamics are approximated accurately:

\[ y \approx y_r \]

- Methods for large-scale applications
  - Goal-oriented model-based reduction
  - Proper orthogonal decomposition (POD)
  - Krylov subspace methods ++
Reduction via Optimization
Willcox et al, 2005

\[
\begin{align*}
\min_{V, x_r} & \quad \frac{1}{2} \int_0^T (y - y_r)^T(y - y_r) \, dt + \frac{\beta}{2} (1 - V_j^T V_j)^2 \\
\text{subject to:} & \\
V^T V \dot{x}_r & = V^T A V x_r + V^T B u \\
y_r & = C V x_r \\
V x_r(0) & = x(0)
\end{align*}
\]
Reduction via Optimization
Willcox et al, 2005

\[
\min_{V, x_r} \frac{1}{2} \int_0^T (y - y_r)^T (y - y_r) dt + \frac{\beta}{2} (1 - V_j^T V_j)^2
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y_r = C V x_r \\
V x_r(0) = x(0)
\]

Error in reduced-order outputs for sample set
Reduction via Optimization

Willcox et al, 2005

\[ \min_{\V, \x_r} \frac{1}{2} \int_0^T (y - y_r)^T (y - y_r) dt + \frac{\beta}{2} (1 - V_j^T V_j)^2 \]

subject to:

\[ V^T V \dot{x}_r = V^T A V x_r + V^T B u \]
\[ y_r = C V x_r \]
\[ V x_r(0) = x(0) \]

Regularization term to yield basis vectors of unit length
Reduction via Optimization
Willcox et al, 2005

\[
\min_{V,x_r} \frac{1}{2} \int_0^T (y - y_r)^T (y - y_r) dt + \frac{\beta}{2} (1 - V_j^T V_j)^2
\]

subject to:

\[
\begin{align*}
V^T V \dot{x}_r &= V^T A V x_r + V^T B u \\
y_r &= C V x_r \\
V x_r(0) &= x(0)
\end{align*}
\]

Reduced output predictions from solution of governing equations
Properties of Reduction Procedure

- Optimized basis targeted to minimize *output* error instead of error on entire domain
  - Important in output feedback optimal control applications
- Model-based optimization: minimizes error in *computed data* (not just *projected data*)
- Tends to preserve important system properties like stability
Proposed Control Scheme

- Generate low-order models from high-fidelity models
- Solve explicit MPC problem for reduced-order model
- Use explicit controller to control plant in output feedback implementation
Numerical Example

• 1D heat equation:

\[
\frac{\partial T}{\partial t}(z,t) = \frac{\partial^2 T}{\partial z^2}(z,t) + u(z,t), \; z \in [0,1] ; \; t > 0
\]

\[
T(0,t) = 0 = T(1,t) \; ; \; t > 0
\]

\[
T(z,0) = 0 \; \; \; \; ; \; z \in [0,1]
\]

• \(u(z,t)\) is a heat source at \(z = 1/3\)
• \(y(z,t)\) is the temperature at \(z = 2/3\)
• Discretize \(\rightarrow\) linear time invariant state space model

\[x \in \mathbb{R}^{200}\]
Model reduction results

\begin{table}
\centering
\begin{tabular}{|c|c|c|}
\hline
$r$ & $H_2^\xi$ for GOMBR & $H_2^\xi$ for POD \\
\hline
1 & 0.6213 & 0.7959 \\
2 & 0.0647 & 0.5023 \\
3 & 0.0230 & 0.0692 \\
4 & 0.0217 & 0.0627 \\
5 & 0.02087 & 0.0841 \\
6 & 0.02085 & 0.0742 \\
7 & 0.0207 & 0.0468 \\
8 & 0.0020 & 0.0020 \\
9 & 0.0012 & 0.0012 \\
10 & $8.6236 \times 10^{-4}$ & $38 \times 10^{-4}$ \\
\hline
\end{tabular}
\end{table}
Closed Loop Evaluation

![Graph showing output and control input over time](image-url)
Future and current work

• Timing and memory requirements
• Demonstrate real-time ability and performance on more complicated problem
• Establish degree of suboptimality
• Invoke robust eMPC versions
• More rigorous model reduction algorithms
  – Error bounds
  – Guarantees on approximation quality
Conclusions

• Introduced a new approach to achieving constrained optimal control for large-scale systems
  – Previously, reduced-order model + MPC

• Complexity reduction on two levels
  – Model reduction reduces the number of states, and facilitates (optimal) control
  – eMPC reduces the on line requirements further; no need for on line optimization