Stability of Model Predictive Control Based on Reduced-Order Models

S. Hovland, C. Løvaas, J. T. Gravdahl and G. C. Goodwin

***Abstract*— In this paper, we present a systematic procedure for obtaining closed-loop stable output-feedback model predictive control based on reduced-order models. The design uses linear state estimators, and applies to open-loop stable systems with hard input- and soft state constraints. Robustness against the model reduction error is obtained by choosing the cost function parameters so as to satisfy a linear matrix inequality condition. We also show by means of an example, that performance is maintained even when the model reduction error is relatively large.**

I. INTRODUCTION

In this paper, we develop a novel approach for achieving exponential stability of model predictive control (MPC) based on reduced order models. The use of model reduction techniques along with MPC is desirable in many applications, in order to reduce the online complexity in implementations that would otherwise run too slowly. In [8] we demonstrated how a significant reduction in complexity could be achieved by truncating only a few number of states, in particular when the MPC horizons are large. The online complexity reduction comes at the cost of introducing an approximation error in the closed-loop system. With the introduction of the approximation error, questions concerning closed-loop stability and feasibility arise. These are very important issues to address, since controllers designed based on reduced-order models might stabilize the reduced-order model and not the plant [11].

Our results hinge on the previous work [14], [15], [16] on robust output-feedback MPC for systems with uncertainties. In this paper we specialize these results to the case of reduced-order models. We ensure stability by choosing the cost function parameters so as to satisfy a set of linear matrix inequality (LMI) conditions, thereby guaranteeing a decreasing Lyapunov function at each time step. To the best of our knowledge, this is the first result that deals systematically with the model reduction error in model predictive control. The results make MPC more attractive for a number of systems that would otherwise be excluded due to the high complexity of the resulting controllers.

In order to guarantee feasibility of the MPC problem, we adopt the soft constraints formulation of [16], in which an additional horizon is introduced to reduce the number of the slack variables. Consequently, the size of the optimization problem is also reduced compared to standard approaches, such as [21]. This extra feature fits nicely into our design, since our goal is to to make our MPC procedure more efficient by introducing reduced-order models.

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The traditional MPC strategy requires significant online computation, limiting the use of this kind of controller to processes with small system state dimension or relatively slow dynamics, since the optimization problem that is solved at each sampling time can otherwise become large. Remedies such as “input blocking”, short horizons etc. are commonly used to reduce the complexity and online computational times. Fast implementation of model predictive control in real-time systems has been considered, among others, by [4] and [20]. Also, it was proposed in [2] to solve multi-parametric quadratic programs (mpQPs) that can be used to obtain explicit solutions to the MPC problem, such that the control input can be efficiently computed by evaluating a piecewise affine function of the system state. Still, as the state dimension and the control horizon and the number of constraints are increased, a large increase in both offline and online complexity follows. The current paper addresses these issues by using reduced-order models.

The paper outline is as follows: In Section II we describe the system formulations that we will consider. The nominal state-feedback design presented in Section III lays the foundation for the reduced-order MPC described in Section IV, where we also prove stability of the procedure and demonstrate performance through a numerical example. Concluding remarks can be found in Section V.

Throughout we use the following notation: $\|x\|_P^2$ denotes x^TPx, $[a, \cdots, c]$ denotes $\begin{bmatrix} a^T & \cdots & c^T \end{bmatrix}^T$ and I_n denotes the $n \times n$ identity matrix.

II. SYSTEM DESCRIPTION

We consider a stable, linear, discrete-time plant, described by the known model

$$x_{k+1}^p = A_p x_k^p + B_p u_k \tag{1a}$$
$$y_k^p = C_p x_k^p, \tag{1b}$$

where $x^p \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ denote the state, input and output, respectively, and the matrices A_p, B_p and C_p are of appropriate dimensions. Here, p denotes the “plant”. The system is subject to the following constraints

$$Vu_k \leq v, \ \forall k \geq 0 \tag{2a}$$
$$Hx_k^p \leq h, \ \forall k \geq 0, \tag{2b}$$

where $V \in \mathbb{R}^{n_v \times m}$, $v \geq 0$, and $H \in \mathbb{R}^{n_h \times n}$.

The input constraints (2a) are "hard", and must be respected at all time, whereas the state constraints (2b) are "soft", and will be treated by penalizing constraint violation in the MPC cost function. This is a natural assumption, since input constraints, such as actuator- and valve limitations are physical limitations that cannot be exceeded. State- and output constraints, on the other hand, often represent "desireables" rather than absolute limitations.

A. REDUCED-ORDER NOMINAL MODEL

The plant model (1) is assumed to be of such a dimension that the online computational requirements conflict with the time available to compute the control input. For the purpose of MPC design, we therefore generate a reduced-order model (ROM), by reducing the order of (1) using an appropriate model reduction technique, such as, for instance, balanced truncation [18], balanced residualization [12] or optimal Hankel norm approximation [1], [3], [10]. These are all rigorous methods with *a priori* error bounds and stability guarantee, provided that (1) is stable. Model reduction techniques are standard textbook material, and good references can, for instance, be found within the robust control literature [23], [19].

The nominal model obtained by model reduction is denoted by

$$x_{k+1} = Ax_k + Bu_k \tag{3a}$$

$$y_k = Cx_k, \tag{3b}$$

where $x \in \mathbb{R}^{n_x}$ such that $n_x < n$, $y \in \mathbb{R}^p$, $A \in \mathbb{R}^{n_x \times n_x}$, $B \in \mathbb{R}^{n_x \times m}$ and $C \in \mathbb{R}^{p \times n_x}$. The nominal model must respect the constraints (2). To enable this, we make the following assumption:

Assumption 1: It is assumed that the constraints (2b) apply to the *outputs* of (1), and consequently apply naturally to the outputs of (3). This can easily be achieved by choosing any states that should be constrained as outputs of the plant.

Remark 1: Associated with the reduced-order model is an approximation error that can be quantified in general terms as follows: When substituting (3) for (1), the *minimum* achievable Hankel norm of the error system is equal to the (n_x+1)-st Hankel singular value of the original system (1) [1], [6], [7]. This error needs to be accounted for in the controller design.

III. NOMINAL CASE WITH STATE FEEDBACK

In this section we present the soft-constrained state-feedback MPC policy proposed in [16] for the nominal system (3), when disregarding the approximation error. The state-feedback policy will subsequently be used in Section IV to develop a robust output-feedback policy for the system (1) based on the reduced-order model (3).

The following optimization problem leads to an MPC scheme with guaranteed nominal stability:

$$\left[P^{N,N_\varepsilon}\right] : J^*(x) = \min_{U,\epsilon,e} J(x,U,\epsilon,e)$$

$$\text{s.t.} \begin{cases} x_0 & = x \\ x_{i+1} & = Ax_i + Bu_i \\ Vu_i & \le v,\ \forall i \in \{0,\cdots,N_u-1\} \\ u_i & = 0,\ \forall i \ge N_u \\ Hx_i & \le h + \epsilon_i,\ \forall i \in \{0,\cdots,N_\epsilon - 1\} \\ Hx_i & \le h + HA^{i-N_\epsilon}e,\ \forall \in i\,\{N_\epsilon,\cdots,N-1\} \\ Tx_N & \le t + TA^{N-N_\epsilon}e, \end{cases} \tag{4}$$

Here, $U = [u_0, \cdots, u_{N_u-1}]$ and $\varepsilon = [\epsilon_0, \cdots, \epsilon_{N_\epsilon-1}]$ are the sequence of N_u inputs and N_ϵ slack variables to be optimized over the horizons N_u and N_ϵ, and $e \in \mathbb{R}^{n_x}$ is an additional vector of slack variables that has been introduced to "summarize" constraint violation beyond the prediction time $i = N_\epsilon - 1$. N is the prediction horizon. Further,

$$J(x,U,\varepsilon,e) \triangleq \|[x,U,\varepsilon,e]\|_P^2 \tag{5}$$

is the cost function, for some appropriate matrix P whose selection will be explained below, and the matrix T and the vector t describe a "terminal constraint set". T and t can e.g. be chosen so that the terminal constraint set equals the maximal output admissible set associated with the state constraints (2b) (see e.g. [5]). We let U^*, ε^* and e^* denote the optimal values of U, ε and e, resulting from $\left[P^{N,N_\varepsilon}\right]$.

Remark 2: Note that by choosing the ingredients in $\left[P^{N,N_\varepsilon}\right]$ in an appropriate way (see [16]), the formulation is equivalent to the standard soft-constrained MPC in [21]. Some special features of our particular formulation is however crucial in our quest for robustly stable MPC based on reduced-order models.

To help describe various conditions on $\left[P^{N,N_\varepsilon}\right]$ and on the cost function matrix P, consider the following autonomous prediction system:

$$\begin{bmatrix} x_{n+1} \\ U_{n+1} \\ \varepsilon_{n+1} \\ e_{n+1} \end{bmatrix} = \underbrace{\begin{bmatrix} A & [B\ 0\ \cdots\ 0] & 0 & 0 \\ 0 & \Gamma(N_u,n_u) & 0 & 0 \\ 0 & 0 & \Gamma(N_\epsilon,n_h) & \bar{H} \\ 0 & 0 & 0 & A \end{bmatrix}}_{\bar{A}_0} \begin{bmatrix} x_n \\ U_n \\ \varepsilon_n \\ e_n \end{bmatrix}, \tag{6}$$

where $\bar{H} \triangleq [0,\cdots,0,H]$, and where $\Gamma\left(\bar{N},\bar{n}\right)$ is a matrix such that, using $\bar{U} = \left[\bar{u}_0,\cdots,\bar{u}_{\bar{N}-1}\right]$, we have $\Gamma\left(\bar{N},\bar{n}\right)\bar{U} = \left[\bar{u}_1,\cdots,\bar{u}_{\bar{N}-1},0\right]$, that is

$$\Gamma\left(\bar{N},\bar{n}\right) = \begin{bmatrix} 0 & I_{\bar{n}} & 0 & \cdots & 0 \\ \vdots & 0 & I_{\bar{n}} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & 0 & \cdots & 0 & I_{\bar{n}} \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \in \mathbb{R}^{\bar{N}\bar{n}\times\bar{N}\bar{n}}. \tag{7}$$

Remark 3: Note that if $N_\epsilon = N$ and P satisfies

$$\bar{A}_0^T P \bar{A}_0 - P + \bar{C}_0^T \operatorname{diag}[Q,R,S]\,\bar{C}_0 = 0, \tag{8}$$

where $\bar{A}_0$ is defined in (6), $Q \in \mathbb{R}^{n_x \times n_x}$, $Q \geq 0$, $R \in \mathbb{R}^{m \times m}$, $R > 0$, $S \in \mathbb{R}^{n_h \times n_h}$, $S > 0$, and where the matrix $\bar{C}_0$ is such that $\bar{C}_0 [x, U, \varepsilon, e] = [x, u_0, \epsilon_0]$, then the cost function (5) satisfies

$$J(x, U, \varepsilon, e) = \|x_{N_u}\|_{P_F}^2 + \sum_{i=0}^{N_u - 1} \left(\|x_i\|_Q^2 + \|u_i\|_R^2 \right) + \|e\|_\Pi^2 + \sum_{i=0}^{N-1} \|\epsilon_i\|_S^2, \quad (9)$$

where $A^T P_F A - P_F = -Q$ and $A^T \Pi A - \Pi = -H^T S H$, and where x_i is given by $\left[P^{N,N_\varepsilon}\right]$ [16].

The state-feedback MPC design proposed in [16] is based on $\left[P^{N,N_\epsilon}\right]$ as follows:

Algorithm 1: (Nominal State-Feedback MPC)

Offline: (i) Choose any integers N, N_u and N_ϵ satisfying $N \geq N_u \geq 1$, $N \geq N_\epsilon \geq 1$.

(ii) Choose any matrices $Q \geq 0$, $R > 0$ and $S > 0$.

(iii) Choose P that satisfies (8).

(iv) Choose any T and t such that the set $X_F \triangleq \{x | Tx \leq t\}$ satisfies

$$Ax \in X_F, \forall x \in X_F, \qquad X_F \subseteq \{x | Hx \leq h\}. \quad (10)$$

Online: At each time step $k \geq 0$, solve $\left[P^{N,N_\varepsilon}\right]$, using $x = x_k$, then apply $u_k = \begin{bmatrix} I & 0 & \cdots & 0 \end{bmatrix} U^*(x)$ to (3).

The following theorem establishes closed-loop stability when applying Algorithm 1 to the nominal system (3), disregarding the "plant" (1) altogether.

Theorem 1: The closed-loop system under Algorithm 1 is globally exponentially stable. Moreover, the closed-loop trajectories satisfy

$$\sum_{k=0}^{\infty} \|x_k\|_Q^2 + \|u_k\|_R^2 + \|\epsilon_k^*\|_S^2 \leq J^*(x_0), \quad (11)$$

where ϵ_k^* denotes the first block component of $\varepsilon^*(x_k)$.

Proof: This is theorem 3 in [16], where the proof can be found. ■

We have now established stability of the MPC design of Algorithm 1, when applied to (3) only. Next, we take model approximation errors into account.

IV. REDUCED-ORDER MPC WITH OUTPUT FEEDBACK

In this section, we propose an output-feedback MPC procedure based on the reduced-order model (3), in which we take into account the error introduced through the model reduction process. We also prove closed-loop stability when applying this controller to the plant (1).

The MPC control input is computed based on the reduced-order state vector x_k at each time step, and x_k should therefore be estimated by an observer based on measurements y_k^p from the plant. For simplicity, we consider a linear estimator of the form

$$\hat{x}_{k+1} = A\hat{x}_k + Bu_k + L\left(y_k^p - C\hat{x}_k\right), \quad (12)$$

where $\hat{x}_k$ denotes the estimated reduced state at time step k, and we choose L such that $(A - LC)$ is Schur (i.e. the eigenvalues lie strictly inside the unit disc).

When uncertainties are taken into account, we will make use of the following matrix function:

$$\Sigma_{\{Q,R,S\}}(P) \triangleq \bar{A}_0^T P \bar{A}_0 - P + \bar{C}_0^T \operatorname{diag}[Q, R, S] \bar{C}_0 \quad (13)$$

The "nominal" cost function matrix, denoted by P_0, is retrieved by solving $\Sigma_{\{Q,R,S\}}(P) = 0$, i.e.

$$\Sigma_{\{Q,R,S\}}(P_0) = 0. \quad (14)$$

Requiring $\Sigma_{\{Q,R,S\}}(P) \leq 0$ implies $P \geq P_0$. We will use $\Sigma_{\{Q,R,S\}}(P)$ at a later stage to search for a P that gives a cost function for the robust case that is an upper bound on the nominal cost.

The proposed output-feedback policy for the system, considering the uncertainties, can now be described as follows:

Algorithm 2: (Output-Feedback MPC)

Offline: (i) Design a state estimator (12).

(ii) Choose any integers N, N_u and N_ϵ satisfying $N \geq N_u \geq 1$, $N \geq N_\epsilon \geq 1$.

(iii) Choose any matrices $Q \geq 0$, $R > 0$ and $S > 0$.

(iv) Choose any matrix P satisfying $\Sigma_{\{Q,R,S\}}(P) \leq 0$.

(v) Choose any T and t such that the set $X_F = \{x | Tx \leq t\}$ satisfies (10).

Online: At each time step $k \geq 0$, solve $\left[P^{N,N_\varepsilon}\right]$ using $x = \hat{x}_k$, then apply $u_k = \begin{bmatrix} I & 0 & \cdots & 0 \end{bmatrix} U^*(\hat{x}_k)$ to (1).

Remark 4: Note that the only difference between Algorithm 2 and Algorithm 1 is the introduction of a state estimator, and the requirement that P satisfies $\Sigma_{\{Q,R,S\}}(P) \leq 0$. Since $\bar{A}_0$ is stable, we can always find such a P.

A. ROBUST STABILITY TEST

Next, following the approach of [16], we propose LMI conditions on the cost function matrix P that are sufficient for closed-loop stability. To this end, we define the augmented state

$$\bar{x} \triangleq [x^p, \hat{x}], \quad (15)$$

where x^p is the plant state and $\hat{x}$ is the estimated ROM state. The dynamics of $\bar{x}$ in closed-loop are described by

$$\bar{x}_{k+1} = \bar{A}\bar{x}_k + \bar{B}\mu_k, \qquad \bar{x}_0 = [x_0, \hat{x}_0] \quad (16)$$

$$\hat{x}_k = \bar{C}\bar{x}_k, \quad (17)$$

where

$$\bar{A} = \begin{bmatrix} A_p & 0 \\ LC_p & A - LC \end{bmatrix} \quad (18)$$

$$\bar{B} = \begin{bmatrix} B_p D_1 \\ BD_1 \end{bmatrix} \quad (19)$$

$$\bar{C} = \begin{bmatrix} 0 & I \end{bmatrix}, \quad (20)$$

and $D_1 = \begin{bmatrix} I & 0 & \cdots & 0 \end{bmatrix}$ is such that $u_k = D_1 \mu_k$, where $\mu_k = [U_k^*, \epsilon_k^*, e_k^*]$ contains the minimizers of $\left[P^{N,N_\varepsilon}\right]$ at time step k. The matrix L is the gain of the state estimator (12).

For the purpose of stability analysis, we need to establish a feasible solution μ_{k+1}^F to $\left[P^{N,N_\varepsilon}\right]$ at time step $k+1$, based on the optimal solution μ_k at the previous time step k. The following lemma establishes the existence such a solution.

Lemma 1: Let $\bar{A}$ and $\bar{B}$ be defined as in (18) and (19). Then

$$F_1 = K_F \begin{bmatrix} LC_p & -LC \end{bmatrix} \tag{21}$$

and

$$F_2 = \begin{bmatrix} \Gamma(N_u, n_u) & 0 & 0 \\ 0 & \Gamma(N_\epsilon, n_h) & \bar{H} \\ 0 & 0 & A \end{bmatrix}, \tag{22}$$

are such that

$$\mu_{k+1}^F = F_1 \bar{x}_k + F_2 \mu_k \tag{23}$$

is a feasible solution to $\left[P^{N,N_\varepsilon}\right]$ at time step $k+1$. Here,

$$K_F = \left[0, H, HA, \cdots, HA^{N_\epsilon - 1}, A^{N_\epsilon}\right], \tag{24}$$

is a particular feasible solution.

Proof: The proof follows from Lemma 5 in [16]. ■

As the final step towards our stability test, we need to find a suitable cost function matrix P. To this end we introduce the following definitions:

$$\Omega(\Omega_0, P) \triangleq \begin{bmatrix} \Omega_0 & 0 \\ 0 & 0 \end{bmatrix} + D_P^T P D_p, \tag{25}$$

with

$$D_P = \begin{bmatrix} \bar{C} & 0 \\ 0 & I_{n_\mu} \end{bmatrix}, \tag{26}$$

and $\Omega_0 \in \mathbb{R}^{(n+n_x)\times(n+n_x)}$.

$$\Phi(\Omega_0, P) \triangleq \begin{bmatrix} \bar{A} & \bar{B} \\ F_1 & F_2 \end{bmatrix} \Omega(\Omega_0, P) \begin{bmatrix} \bar{A} & \bar{B} \\ F_1 & F_2 \end{bmatrix} - \Omega(\Omega_0, P). \tag{27}$$

The stability test for Algorithm 2 can now be stated as follows.

Theorem 2: Assume that, for a given P, there exists a matrix $\Omega_0 \in \mathbb{R}^{(n+n_x)\times(n+n_x)}$ such that,

$$\Omega(\Omega_0, P) > 0 \tag{28a}$$

$$\Phi(\Omega_0, P) < 0, \tag{28b}$$

where $\Omega(\Omega_0, P)$ is as defined in (25) and $\Phi(\Omega_0, P)$ is as defined in (27). Then the closed-loop system under Algorithm 2 is exponentially stable.

Proof: Proving stability follows the well-known path [17] of first showing recursive feasibility, and then showing that there exists a Lyapunov function for the closed-loop system that decreases at each time step. Feasibility at each time step has been established in Lemma 1. Now, consider the Lyapunov function candidate

$$V(\bar{x}, \mu) \triangleq \left\| \begin{bmatrix} \bar{x} \\ \mu \end{bmatrix} \right\|_{\Omega(\Omega_0, P)}^2, \tag{29}$$

which is positive definite in view of (28a), and where μ denotes the minimizers of $\left[P^{N,N_\varepsilon}\right]$, i.e. $\mu_k = [U_k^*, \epsilon_k^*, e_k^*]$. At time step k, we have

$$V_k^* \triangleq V(\bar{x}_k, \mu_k) = \left\| \begin{bmatrix} \bar{x}_k \\ \mu_k \end{bmatrix} \right\|_{\Omega(\Omega_0, P)}^2 \tag{30}$$

$$= \|\bar{x}_k\|_{\Omega_0}^2 + \left\| \begin{bmatrix} \bar{C}\bar{x}_k \\ \mu_k \end{bmatrix} \right\|_P^2 \tag{31}$$

$$= \|\bar{x}_k\|_{\Omega_0}^2 + \left\| \begin{bmatrix} \hat{x}_k \\ \mu_k \end{bmatrix} \right\|_P^2 \tag{32}$$

$$= \|[\bar{x}_k]\|_{\Omega_0}^2 + J_k^*, \tag{33}$$

where $\hat{x}$ takes the place of the nominal state. Similarly, at the next time step $k+1$, the Lyapunov function candidate is given by

$$V_{k+1}^* \triangleq V(\bar{x}_{k+1}, \mu_{k+1}) = \left\| \begin{bmatrix} \bar{x}_{k+1} \\ \mu_{k+1} \end{bmatrix} \right\|_{\Omega(\Omega_0, P)}^2 \tag{34}$$

$$= \|[\bar{x}_{k+1}]\|_{\Omega_0}^2 + J_{k+1}^*. \tag{35}$$

Now μ_{k+1}^F, as in (23), can be used to derive a bound for V_{k+1}^*. Since

$$V_{k+1}^F \triangleq V\left(\bar{x}_{k+1}, \mu^F\right) = \left\| \begin{bmatrix} \bar{x}_{k+1} \\ F_1\bar{x}_k + F_2\mu_k \end{bmatrix} \right\|_{\Omega(\Omega_0, P)}^2 \tag{36}$$

$$= \|\bar{x}_{k+1}\|_{\Omega_0}^2 + \left\| \left[\hat{x}_{k+1}, U_{k+1}^F, \epsilon_{k+1}^F, e_{k+1}^F\right] \right\|_P^2 \tag{37}$$

and

$$V_{k+1}^* = \|[x_{k+1}, \hat{x}_{k+1}]\|_{\Omega_0}^2 + J_{k+1}^*, \tag{38}$$

we have that

$$(\delta V)_{k+1} \triangleq V(\bar{x}_{k+1}, \mu_{k+1}) - V\left(\bar{x}_{k+1}, \mu_{k+1}^F\right) \tag{39}$$

$$= \|\bar{x}_{k+1}\|_{\Omega_0}^2 + J_{k+1}^* - \|\bar{x}_{k+1}\|_{\Omega_0}^2 \tag{40}$$

$$- \left\| \left[\hat{x}_{k+1}, U_{k+1}^F, \epsilon_{k+1}^F, e_{k+1}^F\right] \right\|_P^2$$

$$= J_{k+1}^* - \left\| \left[\hat{x}_{k+1}, U_{k+1}^F, \epsilon_{k+1}^F, e_{k+1}^F\right] \right\|_P^2, \tag{41}$$

and it follows that

$$(\delta V)_{k+1} \leq 0, \tag{42}$$

due to the optimality of J_{k+1}^*, and since μ_{k+1}^F is feasible (but most likely sub-optimal). Obviously, this implies

$$V_{k+1}^* \leq V_{k+1}^F. \tag{43}$$

Now, it remains to show that

$$V_{k+1}^F - V_k^* \leq \alpha \|\bar{x}_k\|^2, \tag{44}$$

for (some arbitrarily small) scalar $\alpha > 0$. For that purpose, we use the property (28b). At time step k, we have

$$\begin{bmatrix} \bar{x}_k \\ \mu_k \end{bmatrix}^T \Phi(\Omega_0, P) \begin{bmatrix} \bar{x}_k \\ \mu_k \end{bmatrix} \tag{45}$$

$$= \begin{bmatrix} \bar{x}_k \\ \mu_k \end{bmatrix}^T \begin{bmatrix} \bar{A} & \bar{B} \\ F_1 & F_2 \end{bmatrix}^T \Omega(\Omega_0, P) \begin{bmatrix} \bar{A} & \bar{B} \\ F_1 & F_2 \end{bmatrix} \begin{bmatrix} \bar{x}_k \\ \mu_k \end{bmatrix} - V_k^*. \tag{46}$$

Now, note that

$$\begin{bmatrix}\bar{A} & \bar{B}\\ F_1 & F_2\end{bmatrix}\begin{bmatrix}\bar{x}_k\\ \mu_k\end{bmatrix} = \begin{bmatrix}\bar{A}\bar{x}_k+\bar{B}\mu_k\\ F_1\bar{x}_k+F_2\mu_k\end{bmatrix} \tag{47}$$

$$= \begin{bmatrix}\bar{x}_{k+1}\\ \mu^F_{k+1}\end{bmatrix}, \tag{48}$$

where μ^F_{k+1} is the feasible solution, as defined in equation (23). By inserting (48) into (46), we have that

$$\begin{bmatrix}\bar{x}_k\\ \mu_k\end{bmatrix}^T \Phi\left(\Omega_0, P\right)\begin{bmatrix}\bar{x}_k\\ \mu_k\end{bmatrix} \tag{49}$$

$$= \begin{bmatrix}\bar{x}_{k+1}\\ \mu^F_{k+1}\end{bmatrix}^T \Omega\left(\Omega_0, P\right)\begin{bmatrix}\bar{x}_{k+1}\\ \mu^F_{k+1}\end{bmatrix} - V^*_k \tag{50}$$

$$= \left\|\begin{bmatrix}\bar{x}_{k+1}\\ \mu^F_{k+1}\end{bmatrix}\right\|_{\Omega(\Omega_0,P)} - V^*_k \tag{51}$$

$$= V^F_{k+1} - V^*_k \tag{52}$$

Since the inequality (28b) is strict it then follows that (44) holds for some $\alpha > 0$. ∎

B. ROBUST DESIGN

Following [16], we next propose a semi-definite program (SDP) that may be used to compute a matrix $P \geq P_0$ that satisfies the stability criterion (28) and is as "close" as possible to the nominal cost function matrix P_0. That is:

$$\inf_{P_1,P_2,\Omega_0} \operatorname{trace}\left(P_1\right) + q\operatorname{trace}\left(P_2\right) \tag{53a}$$

$$\text{s.t.}\begin{cases} P & = \operatorname{diag}\{P_1, P_2\}\\ \Sigma_{\{Q,R,S\}}\left(P\right) & \leq 0\\ \Phi\left(\Omega_0, P\right) & < 0\\ \Omega\left(\Omega_0, P\right) & > 0\end{cases} \tag{53b}$$

where $q > 0$ is a scalar, and where we have also added the structural constraint $P = \operatorname{diag}\{P_1, P_2\}$, such that the cost (5) takes the form $J\left(x, U, \varepsilon, e\right) = \|[x, U]\|^2_{P_1} + \|[\varepsilon, e]\|^2_{P_2}$. Regarding the feasibility of the above SDP, we have the following strong result (which is proven in [9]):

Theorem 3: If the matrices, A_p and $A - LC$, are both stable, then the problem (53) is feasible.

In the sequel, we denote by P^* a feasible and (near) optimal solution to (53).

Remark 5: Since $\Sigma_{\{Q,R,S\}}\left(P^*\right) \leq 0$, we have that $P^* \geq P_0$, where P_0 is as in (14).

By use of $P = P^*$ we obtain the following robust design.

Algorithm 3: (Robust Output-Feedback MPC)

Offline: (i) Choose any integers N, N_u and N_ϵ satisfying $N \geq N_u \geq 1$, $N \geq N_\epsilon \geq 1$.

(ii) Choose any T and t such that the set $X_F = \{x|Tx \leq t\}$ satisfies (10).

(iii) Choose any observer gain such that $A - LC$ is stable.

(iv) Choose any matrices $Q \geq 0$, $R > 0$ and $S > 0$ and determine $P = P^*$ by solving (53).

Online: At each time step $k \geq 0$, solve $\left[P^{N,N_\varepsilon}\right]$ using $x = \hat{x}_k$, then apply $u_k = \begin{bmatrix}I & 0 & \cdots & 0\end{bmatrix} U^*\left(\hat{x}_k\right)$ to (1).

In [9], we address the important question of conservatism of the above robust reduced-order design. Specifically, we show that, under a reasonable assumption, the proposed design is non-conservative in the sense that $P^* \approx P_0$ provided that the neglected dynamics $\Delta(z) \triangleq C_p(zI - A_p)^{-1}B_p - C(zI - A)^{-1}B$ are sufficiently small.

C. NUMERICAL EXAMPLE

We consider a 6th order plant given by

$$A_p = \begin{bmatrix} 0.28 & 0.25 & -0.19 & -0.22 & 0.03 & -0.50\\ 0.25 & -0.47 & 0.30 & 0.17 & -0.11 & -0.11\\ -0.19 & 0.30 & 0.46 & 0.09 & -0.02 & -0.08\\ -0.22 & 0.17 & 0.09 & 0.60 & -0.06 & 0.14\\ 0.03 & -0.11 & -0.02 & -0.06 & 0.46 & -0.13\\ -0.50 & -0.11 & -0.08 & 0.14 & -0.13 & -0.23\end{bmatrix}$$

$B_p = \begin{bmatrix}1.0159 & 0 & 0.5988 & 1.8641 & 0 & -1.2155\end{bmatrix}^T$ and $C_p = \begin{bmatrix}1.2920 & 0 & 0 & 0.2361 & 0.8428 & 0\end{bmatrix}$. The system has a zero at $z = 6.83$, outside the unit circle, and is consequently non-minimum phase. The output y^p_k is subject to soft unit bound constraints, and the input u_k is subject to hard unit bound constraints. We choose $N_u = N = 10$, $N_\epsilon = 2$, $Q = I$, $R = 0.1$ and $S = 1000I$.

First, we reduce the system order from $n = 6$ to $n_x = 5$ and $n_x = 4$ using balanced reduction, and we impose the same constraints on the reduced-order models. Reduced-order models with $n_x = 5$ and $n_x = 4$ leads to model reduction errors $\|\Delta(z)\|_\infty = 6.9885 \times 10^{-6}$ and $\|\Delta(z)\|_\infty = 0.0221$, respectively. The plant is initialized at $x^p_0 = [-0.9044,\ -9.1380,\ -2.5036,\ 0.6696,\ -0.0821,\ -4.0350]$ while the observer is initialized at $\hat{x}_0 = C^+y^p_0$, where C^+ denotes the Moore-Penrose pseudoinverse of C, and y^p_0 is the initial plant output. The SDP (53) is solved using MATLAB with YALMIP [13] and SeDuMi [22].

Fig. 1 compares the closed-loop responses of different robust MPC designs computed using Algorithm 3. The figure also shows the response when using the associated nominal design (NMPC), which is algorithm 3 but using $P = P_0$ as in (14).

For this initial condition, the open-loop response overshoots the upper output constraint by 14%, and so the robust design is good at keeping its soft constraints. Fig. 1 suggests that the robust MPC is not overly conservative when the model uncertainty is relatively small.

If we proceed by truncating to $n_x = 3$, the model reduction error increases by an order of magnitude to $\|\Delta(z)\|_\infty = 0.1373$. In this case, the nominal MPC design fails severely, as illustrated in Fig. 2. In fact, the output for the nominal design oscillates between its soft constraints. On the other hand, the "robustified" design still performs well.

V. CONCLUSIONS

In this paper we have developed a procedure for obtaining closed-loop stability of output-feedback MPC based on reduced-order models. The procedure uses the information available in the original plant model in the offline phase of determining the cost function parameters. Since our main objective is to design an efficient online controller, it is reasonable to put extra work into the offline stage.

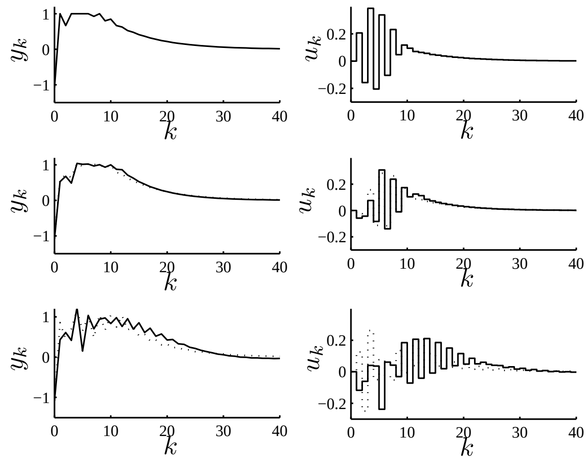

Fig. 1. Top: NMPC using the plant as the nominal model. Center: NMPC (dotted) and robust MPC (solid) using a ROM with $n_x = 5$. Bottom: NMPC (dotted) and robust MPC (solid) using a ROM with $n_x = 4$.

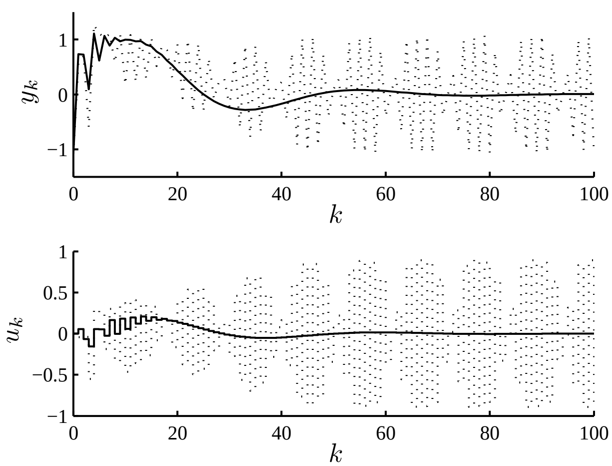

Fig. 2. NMPC (dotted) and robust MPC (solid) using a ROM with $n_x = 3$.

For large-scale systems, this procedure may be too computationally demanding, since we require solving LMIs involving the full system matrices. It seems feasible to further develop the procedure described here by treating parts of the dynamics as model uncertainty.

VI. ACKNOWLEDGMENTS

Hovland, Løvaas and Gravdahl acknowledge the financial support from the Research Council of Norway.

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