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Active surge control using drive torque: dynamic control laws

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Abstract— **In this paper we derive new control laws for a compression system using only drive torque as control variable. First we present, with a minor modification, a recently published stabili ing static control law. Then this control law is extended with a passive part which can be static or dynamic, a robust part using nonlinear damping and an adaptive part. The control laws are module based in the sense that any combination of the control parts can be used in combination with the stabili ing part.**

I. Introduction

Towards low mass flows, the stable operating region of centrifugal compressors is bounded due to the occurrence of surge. This phenomenon is characterized by oscillations in system states such as pressure and mass flow. Surge is undesirable since it introduces the possibility of severe damage to the machine due to vibrations and high thermal loading resulting from lowered efficiency.

Compressor performance is usual described with a compressor map, Fig. 1. This map describes the relation of compressor pressure ratio, mass flow and speed, using constant speed lines in a flow-pressure coordinate system. Surge is considered as an unstable operational mode of the compressor and the stability boundary in the compressor map is called the surge line. This line divides the compressor map in two regions, where the region to the left and right of the surge line corresponds to open loop stable and unstable regions respectively.

Traditionally, surge has been avoided using surge avoidance schemes. Such schemes use various means in order to keep the operating point of the compressor away from the region where surge occurs. Typically, a surge control line is drawn at a distance away from the surge line, leaving a surge margin in the compressor map. The surge avoidance scheme then ensures that the operating point does not cross this line, Fig. 1. This method restricts the operating range of the machine to the region in which the system is open loop stable, and efficiency is limited.

Active surge control is fundamentally different from surge avoidance. In an active surge control scheme the open loop unstable region of the compressor map is sought stabilized with feedback rather than avoided. Thus, the possible operating regime of the machine is enlarged. Active surge control of compressors was first introduced by [1], and since then a number of results have been published. Different actuators have been used and examples include recycle, bleed and throttle valves, gas injection, variable guide vanes and drive torque. For an overview, consult [2], [3] and [4].

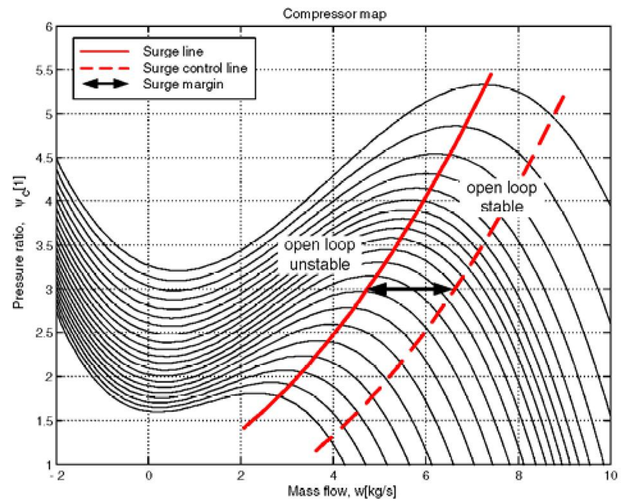

Fig. 1. Compressor performance and surge related definitions

In this paper we derive control laws by only using the drive torque to actively stabilize a compression system. The idea was initially introduced in [5], and further pursued in [6]. The paper is organized as follows. The compression system model is introduced in section II. In section III previous result for drive torque control law is restated and discussed, with a minor extension of the laws previously reported. In section IV the control law is extended with a passive part. Section V addresses the uncertainty involved when cancelling terms in the backstepping procedure, and introduces nonlinear damping in the control law. In section VI the uncertainty in cancellation is pursued further with adaption of constant parameters. Simulations are presented in section VII, and section VIII gives some concluding remarks.

II. Compression system model

A classical result in the field of compressor surge modeling is the model of Greitzer [7], which covers a basic compression system consisting of a compressor, a plenum volume, in-between ducting and a throttle valve as shown in Fig. 2. In [8] the authors extended the Greitzer model to also incorporate variable impeller speed. A similar model was derived in [9] using an approach based on energy analysis, which is the model used here.

Consider a compression system in which a centrifugal compressor supplies compressed gas to a duct which discharges into a plenum volume, from which the compressed gas discharges over a throttle. A model for this system can be written as

$$\dot{\ } = \frac{a_0^2}{V}\left(w - w_t\left(\ \right)\right) \quad (1)$$

$$\dot{w} = \frac{A}{L}\left(\ _c\left(w, \omega\right) - \ \right) \quad (2)$$

$$\dot{\omega} = \frac{1}{J}\left(\tau_d - \tau_c\left(w, \omega\right)\right). \quad (3)$$

where is the plenum pressure, w is the duct mass flow and ω is impeller speed. Throttle mass flow is given by $w_t(\ )$, total pressure downstream compressor is denoted $_c(w, \omega)$ and torque experienced by impeller due to compressor fluid flow is given by $\tau_c(w, \omega)$. Furthermore, the various constants represent speed of sound at ambient conditions a_0, volume of plenum V, cross section of duct A, length of duct L and inertia of rotating parts J.

Pressure dynamics is derived by computing the mass balance of the plenum volume assuming isentropic conditions and uniform pressure. Mass flow dynamics is derived by computing the momentum balance of the duct connecting compressor and plenum assuming incompressible one dimensional flow in the duct and compressor. Moreover, dynamic effects related to the compressor stage are assumed small, leaving total pressure downstream the compressor as a pure mapping from mass flow and impeller speed. Impeller speed dynamics is derived by calculating the angular momentum balance. This is where the drive torque appears, assumed to be at our disposal as system input.

Models for throttle mass flow and compressor torque are taken as

$$w_t\left(\ \right) = k_t sign\left(\ - \ _0\right)\sqrt{\left|\ - \ _0\right|} \quad (4)$$

$$\tau_c\left(w, \omega\right) = k_c \left|w\right| \omega, \quad (5)$$

where $_0$ is ambient constant pressure, k_t is the throttle constant (proportional to throttle opening) and k_c is a compressor torque constant. The model of throttle valve mass flow is slightly modified relative to [9]. The modification involves including the possibility of negative mass flow through the valve, and to this end the throttle characteristic is assumed to be symmetric.

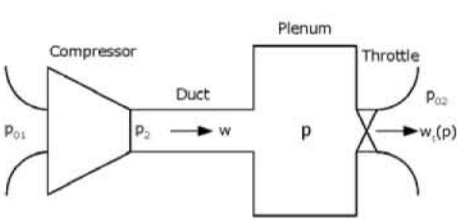

Fig. 2. Compression system

From (1)-(3) it can be seen that a desired equilibrium satisfy

$$w^e = k_t\sqrt{\ ^e - \ _0} \quad (6)$$

$$\ ^e = \ _c\left(w^e, \omega^e\right) \quad (7)$$

$$\tau_d^e = k_c w^e \omega^e \quad (8)$$

where in all practical cases the desired equilibrium involves positive valued states and a higher plenum than ambient pressure. For computational convenience of subsequent analysis, the desired equilibrium is shifted to the origin

$$x_1 = \ - \ ^e \quad x_2 = w - w^e \quad x_3 = \omega - \omega^e \quad u = \tau_d - \tau_d^e \quad (9)$$

where superscript e refers to desired equilibrium. The model is then rewritten in error coordinates using (1)-(3) and (9)

$$\dot{x}_1 = k_1\left(x_2 + f_1\left(x_1\right)\right) \quad (10)$$

$$\dot{x}_2 = k_2\left(f_2\left(x_2, x_3\right) - x_1\right) \quad (11)$$

$$\dot{x}_3 = k_3\left(u + f_3\left(x_2, x_3\right)\right) \quad (12)$$

where $k_1 = \frac{a_0^2}{V}$, $k_2 = \frac{A}{L}$, $k_3 = \frac{1}{J}$ and

$$f_1\left(x_1\right) = w^e - w_t\left(x_1 + \ ^e\right) \quad (13)$$

$$f_2\left(x_2, x_3\right) = \ _c\left(x_2 + w^e, x_3 + \omega^e\right) - \ ^e \quad (14)$$

$$f_3\left(x_2, x_3\right) = \tau_d^e - \tau_c\left(x_2 + w^e, x_3 + \omega^e\right). \quad (15)$$

Moreover, from (4), (5), (13), (14) and the observations of pressure downstream the compressor being strictly increasing in impeller speed, the various functions can be shown to have the properties

$$(a-b)\left(f_1(a) - f_1(b)\right) < 0 \quad (16)$$

$$(a-b)\left(f_2(x_2, a) - f_2(x_2, b)\right) > 0 \quad (17)$$

$$(a-b)\left(f_3(x_2, a) - f_3(x_2, b)\right) \leq 0 \quad (18)$$

and $f_1(0) = f_2(0,0) = f_3(0,0) = 0$.

III. State feedback backstepping design

Control laws to be derived are based on results from [6]. These results are therefore restated for convenience and completeness, as well as the basis for a minor modification. The backstepping procedure resulted in backstepping error variables

$$z_1 = x_1 \quad z_2 = x_2 \quad z_3 = x_3 - \alpha_3\left(z_2\right) \quad (19)$$

where $\alpha_3(z_2)$ is a stabilizing function defined by

$$\alpha_3\left(z_2\right) = -c_3 z_2 \quad (20)$$

where c_3 is a positive constant chosen according to

$$c_3 \geq \frac{\frac{\partial f_2(z_2, \alpha_3)}{\partial z_2} + \delta_3}{\frac{\partial f_2(z_2, \alpha_3)}{\partial \alpha_3}}. \quad (21)$$

Furthermore, the input $u = u_1 + u_2$ with

$$u_1 = -c_4 z_3 \quad (22)$$

and the function

$$V_3\left(\mathbf{z}\right) = \frac{1}{d_3 k_3}\left(\frac{1}{2}d_1 z_1^2 + \frac{1}{2}d_2 z_2^2 + \frac{1}{2}d_3 z_3^2\right) \quad (23)$$

resulted in

$$\begin{aligned}\dot{V}_3(t,\mathbf{z}) \leq\ & \frac{d_2k_2}{d_3k_3} z_1 f_1(z_1) - \mathbf{z}_{2,3}^T Q(t)\,\mathbf{z}_{2,3} \\ & + z_3(u_2 + f_3(z_2, z_3+\alpha_3) \\ & + c_3 \frac{k_2}{k_3}\left(f_2(z_2, z_3+\alpha_3) - z_1\right)) \end{aligned} \quad (24)$$

where $\mathbf{z} = (z_1,\ z_2,\ z_3)$, $\mathbf{z}_{2,3} = (z_2,\ z_3)$, d_i and c_4 are positive constants and

$$Q(t) = \begin{bmatrix} \frac{d_2k_2}{d_3k_3}\delta_3 & -\frac{1}{2}\frac{d_2k_2}{d_3k_3}\frac{\partial f_2(r_1,r_2)}{\partial r_2} \\ -\frac{1}{2}\frac{d_2k_2}{d_3k_3}\frac{\partial f_2(r_1,r_2)}{\partial r_2} & c_4 \end{bmatrix} \quad (25)$$

with $r_1 = z_2$ and $r_2 \in L(z_3+\alpha_3, \alpha_3)$ [1].

An alternative formulation for compressor torque in (24) relative to [6] can be made by rewriting

$$f_3(z_2, z_3+\alpha_3) = f_3(z_2, x_3) - f_3(z_2,\alpha_3) + f_3(z_2,\alpha_3) \quad (26)$$

where (19) has been used. The inequality (24) can now be expressed

$$\begin{aligned}\dot{V}_3(\mathbf{z}) \leq\ & \frac{d_2k_2}{d_3k_3} z_1 f_1(z_1) - \mathbf{z}_{2,3}^T Q(t)\,\mathbf{z}_{2,3} \\ & + z_3(u_2 + f_3(z_2, \alpha_3) \\ & + c_3 \frac{k_2}{k_3}\left(f_2(z_2, z_3+\alpha_3) - z_1\right)) \end{aligned} \quad (27)$$

where it has been used that $z_3(f_3(z_2,x_3) - f_3(z_2,\alpha_3)) \leq 0$ due to (18) and (19).

Two implementations of u_2 where proposed in [6]. One in which all terms enclosed by the same bracket as u_2 in (24) where cancelled, and one that did not cancel $f_3(z_2, z_3+\alpha_3)$. One additional implementation can be made based on the rewriting (26), where all terms enclosed by the same bracket as u_2 in (27) is cancelled. The three variants of the control law, $u = u_1 + u_2$, are summarized for convenience

$$\begin{aligned} u_2^a &= -f_3(z_2, z_3+\alpha_3) && -c_3\tfrac{k_2}{k_3}\left(f_2(z_2,z_3+\alpha_3) - z_1\right) \\ u_2^b &= -f_3(z_2,\alpha_3) && -c_3\tfrac{k_2}{k_3}\left(f_2(z_2,z_3+\alpha_3) - z_1\right) \\ u_2^c &= && -c_3\tfrac{k_2}{k_3}\left(f_2(z_2,z_3+\alpha_3) - z_1\right) \end{aligned} \quad (28)$$

where u_1, given by (22), is unchanged for the various implementations of u_2. Stability results are derived using (23) and

$$\dot{V}_3^{a,b,c}(\mathbf{z}) \leq -\mathbf{z}_{2,3}^T Q^{a,b,c}(t)\,\mathbf{z}_{2,3} + \frac{d_2k_2}{d_3k_3} z_1 f_1(z_1) \quad (29)$$

where $Q^a(t) = Q^b(t) = Q(t)$, $Q^c(t)$ contains some additional terms relative to $Q(t)$ and the superscript refers to the control law in question. Semi global asymptotic stability of $\mathbf{z} = \mathbf{0}$ was concluded using (23), (29) and (16), when (21) and $Q^{a,b,c}(t) > 0$ holds semi globally. Moreover, if

$$z_1 f_1(z_1) \leq -\delta_1 z_1^2 \quad (30)$$

the results are exponential since (23) and (29) are quadratically bounded in $\|z\|_2$. This inequality holds semi globally for the throttle model (4). It does not hold globally since $\lim_{|p|\to\infty} \frac{\partial f_1(x_1)}{\partial x_1} = 0$.

For all three control laws, the gain c_3 has a lower bound given by (21). For control laws a and b it is sufficient to choose c_4 arbitrarily large, whereas it must be chosen sufficiently large for control law c. The differences of the control laws are found in the way compressor torque is handled. The first law directly cancels the model for compressor torque and the second cancels parts of it. The third control law does not include an explicit term for cancelling compressor torque, here stability is achieved by dominating torque with a linear term (and hence require c_4 sufficiently large).

Remark 1: Control law b has an additional stabilizing term in (29), $z_3(f_3(z_2,x_3) - f_3(z_2,\alpha_3))$, compared to control law a. Comparing a and b, is seems like the cancellation done in b is more robust than the one done in a since it leaves an additional stabilizing term. This claim has no rigid justification in analysis, but is rather based the observation that $\dot{V}_3$ becomes "more negative" for control law b than for a. The additional stabilizing term can also be shown for control law c, but in this case c_4 is required sufficiently large in contrast to requirement of arbitrarily large in the first two cases.

IV. PASSIVITY

The derived control laws are all on the form $u = u_1 + u_2$. Recalling the analysis leading to (24), it can be recognized that redefining $u = u_1 + u_2 + u_3$ leaves a term z_3u_3 in (29). Since u_3 is at our disposal and can be chosen freely, this motivates the investigation of passivity properties of the pair z_3u_3.

Using the redefined control input with previously derived u_1 and u_2, the general system dynamics can be expressed as

$$\Sigma_z : \begin{cases} \dot{\mathbf{z}} = \mathbf{f}_z(\mathbf{z}, u_3) \\ y_z = z_3 \end{cases} \quad (31)$$

where u_3 and y_z are system input and output respectively. Dynamics of this system will depend on the specific implementation of u_2, (28). Furthermore, it follows from previous analysis that $\mathbf{z} = \mathbf{0}$ is semi-globally asymptotically stable for $\dot{\mathbf{z}} = \mathbf{f}_z(\mathbf{z}, 0)$.

The approach of investigating passivity properties was motivated by the appearance of z_3u_3 in (24). Hence, a storage function, V_{s,Σ_z}, for the system is defined on the basis of V_3

$$V_{s,\Sigma_z}(z) = V_3(z). \quad (32)$$

Using results from previous analysis, the time derivative of (32) along the solution of (31) results in the inequality

$$z_3u_3 \geq \dot{V}_{s,\Sigma_z}^{a,b,c}(\mathbf{z}) + z_{2,3}^T Q^{a,b,c}(t)\, z_{2,3} - \frac{d_2k_2}{d_3k_3} z_1 f_1(z_1) \quad (33)$$

where it has been shown that $Q^{a,b,c}(t)$ are positive definite by choosing the control gains properly and $-\frac{d_2k_2}{d_3k_3} z_1 f_1(z_1)$ is positive definite by (16). Rewriting the inequalities as $z_3u_3 \geq \dot{V}_{s,\Sigma_z}^{a,b,c}(\mathbf{z})$, it is concluded that $\mathbf{\Sigma}_z$ is passive, [10]. To show that $\mathbf{\Sigma}_z$ is output strictly passive, the gain c_4 is temporary redefined as $c_4 = c_4' + \delta_{c_4}$ for some $\delta_{c_4} > 0$.

[1] r_2 is some point on the line segment joining $z_3 + \alpha_3$ and α_3

From this definition, c_4' is considered used in (25) and δ_{c_4} is used to show the system output strictly passive. Using the temporary redefined c_4, inequalities can be rewritten $z_3 u_3 \geq \dot{V}^{a,b,c}_{s,\Sigma_z}(\mathbf{z}) + \delta_{c_4} z_3^2$, by which it is concluded that Σ_z is output strictly passive. Furthermore, from (33) and recognizing that $\mathbf{z}_{2,3}^T Q^{a,b,c}(t)\,\mathbf{z}_{2,3} - \frac{d_2 k_2}{d_3 k_3} z_1 f_1(z_1)$ is positive definite in $\mathbf{z}$, it is concluded that Σ_z is strictly passive.

Due to Σ_z being output strictly passive of the given form, it is also finite gain L_2 stable with finite-gain less than $\frac{1}{\delta_{c_4}}$.

It is known from general theory of passivity that a strictly passive system in feedback interconnection with a strictly passive or a output strictly passive and zero state observable system, poses asymptotic stability of the overall system origin. This motivates the definition of a second system

$$\Sigma_{u3} : \begin{cases} \dot{\mathbf{z}}_{u3} = \mathbf{f}_{u3}(\mathbf{z}_{u3}, z_3) \\ y_{u3} = -u_3 \end{cases} \quad (34)$$

representing a SISO dynamic control action for u_3. Notice that system input is a linear combination of equilibrium error variables for compressor mass flow and impeller speed ($z_3 = x_3 + c_3 x_2$), and system output is at our disposal. Hence, the system Σ_{u3} can be chosen under restriction of a given input and output pair (dynamics of the system can be chosen freely). Choosing it to comply with the properties of strictly passive or output strictly passive and zero state observable will guarantee asymptotic stability for overall feedback interconnection with Σ_z. Such a dynamic system can for instance be a PID type control law, with parameters chosen such that it represents a strictly passive system, [11]. Moreover, it is believed that best results for the control law is obtained by choosing c_3 and c_4 as small as possible so that stability is achieved for Σ_z, and then "tune" the dynamic part, Σ_{u3}, of the controller for performance. The motivation for this is found in disturbances, since a dynamic controller can offer low pass filtering in addition to gain. Furthermore, a dynamic control law can also offer integrating effect on its error variable, which can improve steady state performance.

Assume now that Σ_{u3} is chosen according to preceding discussion. An immediate consequence is the passivity of overall feedback interconnection of Σ_z and Σ_{u3}. If system Σ_{u3} is output strictly passive, it follows that the interconnection is finite gain L_2 stable. Furthermore, if (30) holds and Σ_{u3} is linear, the result will be exponential stability of the overall system.

Sofar the overall stability of Σ_z in feedback interconnection with a second dynamic system of appropriate passive properties have been discussed. owever, there also exists similar results stating that the origin of Σ_z in feedback interconnection with a time-varying memoryless function $y_{u3} = h_{u3}(t, z_3)$, where $h_{u3}(t, z_3)$ is passive, remains asymptotically stable, [10]. One possible choice is a saturated linear function $h_{u3}(t, z_3) = sat(cz_3)$. This can be used to practically replace u_1 for implementations a and b, by choosing c_4 vanishingly small.

V. Uncertainty

The control law involves direct cancellation of model terms. These cancellations can be found in u_2, and are done in the final step of the design procedure to avoid sign indefinite cross terms in (29). Since the terms implemented in u_2 are model based, these cancellation will involve some uncertainty with respect to the actual value. Following [12], a relationship between model and actual value will be taken as

$$f_{actual} = f_{model}(1 + \Delta(t)) \quad (35)$$

where $\Delta(t)$ is assumed bounded. From this it can be recognized that $\Delta(t)$ represents the models deviation from its actual value, with $\Delta(t) = 0$ implying perfect match of model and real value, and the model error increases as $\Delta(t)$ deviates away from zero.

In order to limit the effect of errors introduced by uncertainty in cancelling terms, the control law is extended with a nonlinear damping term u_4. This is not introduced to achieve asymptotic stability in the presence of uncertainty, but rather limit the effect of these errors and to guarantee bounded solutions in their presence.

We restrict the analysis to the case where (34) is strictly passive with a radially unbounded storage function $V_{s,u3}(\mathbf{z}_{u3})$ satisfying

$$-u_3 z_3 \geq \dot{V}_{s,u3}(\mathbf{z}_{u3}) + W_{s,u3}(\mathbf{z}_{u3}),\ W_{s,u3}(\mathbf{z}_{u3}) > 0 \quad (36)$$

where $W_{s,u3}(\mathbf{z}_{u3})$ is radially unbounded.

Analysis is now conducted for the control law including the passive part u_3, since this gives the most general case (also incorporating the case when no passive control part is used). Consider the function

$$V_\Delta(\mathbf{z}, \mathbf{z}_{u3}) = V_3(\mathbf{z}) + V_{s,u3}(\mathbf{z}_{u3}) \quad (37)$$

The overall control law is now defined by $u = u_1 + u_2 + u_3 + u_4$ (including also the passive part) or $u = u_1 + u_2 + u_4$ (excluding passive part), where u_4 constitutes the nonlinear damping part of the control. Using the previous result (24) or (27), (28), (29) and (36) with the uncertainty (35), the time derivative of (37) is upper bounded by

$$\begin{aligned} \dot{V}^{a,b,c}_\Delta(t, \mathbf{z}, \mathbf{z}_{u_3}) \leq\ & -\mathbf{z}_{2,3}^T Q^{a,b,c}(t)\,\mathbf{z}_{2,3} + \frac{d_2 k_2}{d_3 k_3} z_1 f_1(z_1) \\ & - W_{u3}(\mathbf{z}_{u3}) \\ & + z_3(u_4 + f_{\Delta 1}(\mathbf{z})\Delta_1(t) \\ & + f_{\Delta 2}(\mathbf{z})\Delta_2(t)) \end{aligned} \quad (38)$$

where $\Delta_i(t)$'s are the $\Delta(t)$ defined in (35) for each cancellation and the f_Δ's are given by

$$f_{\Delta 1}(\mathbf{z}) = c_3 \frac{k_2}{k_3}(f_2(z_2, z_3 + \alpha_3) - z_1) \quad (39)$$

$$f^{a,b,c}_{\Delta 2}(z_2, z_3) = \{f_3(z_2, z_3 + \alpha_3), f_3(z_2, \alpha_3), 0\} \quad (40)$$

where the function (40) will vary with the different implementations of u_2, (28). The terms $f_{\Delta i}\Delta_i(t)$ represent error made when using a given model.

The effect of perturbation resulting from uncertainty is now sought reduced with the control input

$$u_4 = -\kappa_1 z_3 f_{\Delta 1}^2 - \kappa_2 z_3 f_{\Delta 2}^2 \tag{41}$$

and an upper bound for (38) is then given by

$$\dot{V}_{\Delta}^{a,b,c}(t, \mathbf{z}, \mathbf{z}_{u_3}) \leq -W_{\Delta}(\mathbf{z}, \mathbf{z}_{u_3}) + \|\Delta_{sum}(t)\| \tag{42}$$

where $W_{\Delta}(\mathbf{z}, \mathbf{z}_{u_3}) = -\mathbf{z}_{2,3}^T Q^{a,b,c}(t)\,\mathbf{z}_{2,3} + \frac{d_2 k_2}{d_3 k_3} z_1 f_1(z_1) - W_{u3}(\mathbf{z}_{u3})$ and $\Delta_{sum}(t) = \frac{1}{4\kappa_1}\Delta_1^2(t) + \frac{1}{4\kappa_2}\Delta_2^2(t)$ is scalar and time varying, [12]. Combining the previous results of positive definite $Q^{a,b,c}$ and the assumption made in (36), it is known that $W_{\Delta}(\mathbf{z}, \mathbf{z}_{u3})$ is positive definite in its arguments semi globally. Furthermore, it is noticed that $\|\Delta_{sum}(t)\|$ reduces with increasing κ's. Since $V_{\Delta}(\mathbf{z}, \mathbf{z}_{u3})$ and $W_{\Delta}(\mathbf{z}, \mathbf{z}_{u3})$ are positive definite and radially unbounded functions, they can be bounded by class $\mathcal{K}_{\infty}$ functions as $\gamma_1(\|(\mathbf{z}, \mathbf{z}_{u3})\|) \leq V_{\Delta}(\mathbf{z}, \mathbf{z}_{u3}) \leq \gamma_2(\|(\mathbf{z}, \mathbf{z}_{u3})\|)$ and $\gamma_3(\|(\mathbf{z}, \mathbf{z}_{u3})\|) \leq W_{\Delta}(\mathbf{z}, \mathbf{z}_{u3})$. It can now be shown that system states are bounded by

$$\|(\mathbf{z}, \mathbf{z}_{u3})\|_{\infty} \leq \max\left\{\begin{array}{c} \gamma_1^{-1} \circ \gamma_2 \circ \gamma_3^{-1}(\|\Delta_{sum}(t)\|_{\infty}), \\ \gamma_1^{-1} \circ \gamma_2(\|(\mathbf{z}(t_0), \mathbf{z}_{u3}(t_0))\|) \end{array}\right\} \tag{43}$$

which gives a condition of worst case with respect to effects of model uncertainty or initial conditions , [12]. Furthermore, it can be shown that the overall system is input-to-state stable with respect to the disturbance $\Delta_{sum}(t)$. Moreover, system states convergence to the set

$$\Omega_{\Delta} = \left\{\|(\mathbf{z}, \mathbf{z}_{u3})\| \leq \gamma_1^{-1} \circ \gamma_2 \circ \gamma_3^{-1}(\|\Delta_{sum}(t)\|_{\infty})\right\} \tag{44}$$

and semi global uniform bounded solutions $(\mathbf{z}(t), \mathbf{z}_{u3}(t))$ follows.

As already mentioned, these results holds whether or not the passive control part is used. The practical difference lies in the estimates of the global bound, since the γ's will be different for the two cases. owever, the $\Delta_{sum}(t)$ will be the same whether or not the passive part is included, since $\Delta_{sum}(t)$ is related to u_2 only. Furthermore, these results also give an estimate on bounds of solutions, (43). This can be used to estimate a less restrictive bound on gains used to achieve the semi global results, by restricting the region of attraction from semi global to some predefined region for a specific plant.

The nonlinear damping was introduced to guarantee a bound of solutions in presence of uncertainty in cancelled terms. The results shows that solutions are bounded and will converge to (44). Furthermore, it can be recognized from (42) that $\Delta_{sum}(t) = 0$ results in $\dot{V}_{\Delta}(t, \mathbf{z}, \mathbf{z}_{u3}) \leq -W_{\Delta}(\mathbf{z}, \mathbf{z}_{u3})$, and the system is asymptotically stable. Tuning of control gains will depend on both the specific implementation of (28) and the passive part of the controller, since these factors have an effect on the $f_{\Delta 2}$ and γ functions, which in turn defines Ω.

VI. State feedback adaptive backstepping design

The state feedback backstepping procedure resulted in three basic control laws (28), where their differences is due to how they cancel the compressor torque. It has also been shown that these basic control laws can be extended with a passive part, $z_3 \mapsto -u_3$. Furthermore, using nonlinear damping guaranteed bounded solutions in presence of bounded uncertainties in cancelled terms.

In this section the uncertainty involved in cancelling terms are sought improved by parameter estimation. More specifically, constant parameters appearing affine in the cancelled terms will be addressed. All cancellations comply with the matching condition, which means that cancelled terms are in the span of the control variable. More specifically, all cancellations are done in the final step of the procedure and collected in the u_2 part of control law u.

Three model specific constants, k_c, k_2 and k_3, can be recognized from (28) (with k_c incorporated in f_3). owever, since k_2 and k_3 always appear in the configuration $\frac{k_2}{k_3}$, these will be treated as one unknown constant only. For analysis it is convenient to rewrite $f_3(q_1, q_2) = \tau_d^e - k_c f_3'(q_1, q_2)$, where $f_3'(q_1, q_2) = |q_1 + w^e|(q_2 + \omega^e)$, in order to get an expression where k_c appears explicit. At this stage we ignore that τ_d^e contains k_c, since in the actual implementation we have $\tau_d = \tau_d^e + u$ and the τ_d^e's of f_3 and u will cancel out. For the same reason we express $u_2^c = -\tau_d^e + k_c w^e \omega^e - c_3 \frac{k_2}{k_3}(f_2(z_2, z_3 + \alpha_3) - z_1)$.

Parameter estimators are derived by certainty equivalence. This involves replacing constants in the previously derived control laws by their estimates, and then analyze to design the dynamic part for update laws. Parameter estimates will generally be denoted by θ, with subscript 1 referring to $\frac{k_2}{k_3}$ and subscript 2 referring to k_c. Furthermore, deviation of parameter estimates from their actual value is denoted

$$z_{\theta} = k_{\theta} - \theta \tag{45}$$

with k_{θ} referring to the actual constant. The details of the derivation is only shown for u_2^a, when deriving for the two other will consist of the exact same exercise. Consider the function

$$V_{\theta}(\mathbf{z}, \mathbf{z}_{\theta}) = V_3(\mathbf{z}) + \frac{1}{2c_{\theta 1}} z_{\theta 1}^2 + \frac{1}{2c_{\theta 2}} z_{\theta 2}^2 \tag{46}$$

which is positive definite and radially unbounded in $(\mathbf{z}, \mathbf{z}_{\theta})$, where $\mathbf{z}_{\theta} = (z_{\theta 1}, z_{\theta 2})$. The time derivative of (46) using the certainty equivalence control law

$$u_{2\theta}^a = -\tau_d^e + \theta_2 f_3'(z_2, z_3 + \alpha_3) - c_3\theta_1(f_2(z_2, z_3 + \alpha_3) - z_1) \tag{47}$$

(45) and (27), can be upper bounded by

$$\begin{aligned} \dot{V}_{\theta}^a(\mathbf{z}, \mathbf{z}_{\theta}) \leq\ & \frac{d_2 k_2}{d_3 k_3} z_1 f_1(z_1) - \mathbf{z}_{2,3}^T Q^a(t)\,\mathbf{z}_{2,3} \\ & + z_{\theta 1}^a\left(\frac{1}{c_{\theta 1}}\dot{z}_{\theta 1} + z_3 c_3(f_2(z_2, z_3 + \alpha_3) - z_1)\right) \\ & + z_{\theta 2}^a\left(\frac{1}{c_{\theta 2}}\dot{z}_{\theta 2} - z_3 f_3'(z_2, z_3 + \alpha_3)\right) \end{aligned} \tag{48}$$

where $\dot{z}_{\theta 1}$ and $\dot{z}_{\theta 2}$ are at our disposal. These are now chosen such that the content their brackets in (48) becomes zero

$$\dot{z}_{\theta 1}^a = -c_{\theta 1} c_3 z_3 (f_2(z_2, z_3 + \alpha_3) - z_1) \tag{49}$$

$$\dot{z}_{\theta 2}^a = c_{\theta 2} z_3 f_3'(z_2, z_3 + \alpha_3) \tag{50}$$

which results in

$$\dot{V}_\theta^a(\mathbf{z},\mathbf{z}_\theta) \le \frac{d_2k_2}{d_3k_3} z_1 f_1(z_1) - \mathbf{z}_{2,3}^T Q^a(t)\,\mathbf{z}_{2,3} \quad (51)$$

being equal to (29). Using previous results, it is known that $\dot{V}_\theta(\mathbf{z},\mathbf{z}_\theta) \le -W(\mathbf{z})$ where $W(\mathbf{z}) = \frac{d_2k_2}{d_3k_3} z_1 f_1(z_1) - \mathbf{z}_{2,3}^T Q^a(t)\,\mathbf{z}_{2,3}$ is positive definite and radially unbounded in $\mathbf{z}$. This implies that all solutions $(\mathbf{z}(t), \mathbf{z}_\theta(t))$ are uniformly bounded and that $\lim_{t\to\infty}\mathbf{z}\to\mathbf{0}$, since $\lim_{t\to\infty}W(\mathbf{z})\to 0$. owever, no conclusion can be drawn on the convergence of parameter estimates.

Following the same approach it can be verified that the other two control laws are given by $u_{2\theta}^b = -\tau_d^e + \theta_2 f_3'(z_2,\alpha_3) - c_3\theta_1(f_2(z_2, z_3+\alpha_3) - z_1)$ and $u_{2\theta}^c = -\tau_d^e + \theta_2 w^e\omega^e - c_3\theta_1(f_2(z_2, z_3+\alpha_3) - z_1)$. For both of these, the update law for θ_1 will be the same as that for $u_{2\theta}^a$, (49). The update law for θ_2 will be $\dot{z}_{\theta 2}^b = c_{\theta 2} z_3 f_3'(z_2,\alpha_3)$ and $\dot{z}_{\theta 2}^c = c_{\theta 2} z_3 w^e\omega^e$ respectively. Furthermore, it is considered a design choice whether c_3 is included as part of the unknown parameter θ_1 or not.

An alternative formulation can be made by splitting up the brackets enclosing f_2 and z_1 in (28), and then treat the constant $\frac{k_2}{k_3}$ as two different constants when appearing affine in f_2 and z_1 respectively. Treating them as different constants has no root in the physical model, but is included to investigate influence on overall system dynamics. A certainty equivalence control law is then given by

$$u_2^a = -\tau_d^e + \theta_2 f_3'(z_2, z_3+\alpha_3) + c_3\theta_{11} z_1 - c_3\theta_{12} f_2(z_2, z_3+\alpha_3) \quad (52)$$

where the notation θ_{11} and θ_{12} is used for estimates of $\frac{k_2}{k_3}$ appearing in front of z_1 and f_2 respectively. The result for this scheme is derived in same manner as that of (47). Consider the function

$$V_\theta(\mathbf{z},\mathbf{z}_\theta) = V_3(\mathbf{z}) + \frac{1}{2c_{\theta 11}} z_{\theta 11} + \frac{1}{2c_{\theta 12}} z_{\theta 12} + \frac{1}{2c_{\theta 2}} z_{\theta 2} \quad (53)$$

and the certainty equivalence control law of (52). The time derivative of (53) is found as

$$\begin{aligned}\dot{V}_\theta^a(\mathbf{z},\mathbf{z}_\theta) \le\ & \frac{d_2k_2}{d_3k_3} z_1 f_1(z_1) - z_{2,3}^T Q^b(t)\, z_{2,3} \\ & + \left(\frac{1}{c_{\theta 11}}\dot{z}_{\theta 11} - c_3 z_3 z_1\right) z_{\theta 11} \\ & + \left(\frac{1}{c_{\theta 12}}\dot{z}_{\theta 12} + c_3 z_3 f_2(z_2, z_3+\alpha_3)\right) z_{\theta 12} \\ & + \left(\frac{1}{c_{\theta 2}}\dot{z}_{\theta 2} - z_3 f_3'(z_2, z_3+\alpha_3)\right) z_{\theta 2} \quad (54)\end{aligned}$$

where $\dot{z}_{\theta 11}$, $\dot{z}_{\theta 12}$ and $\dot{z}_{\theta 2}$ are at our disposal. These are now chosen such that content their brackets becomes zero

$$\dot{z}_{\theta 11}^a = c_{\theta 11} c_3 z_3 z_1 \quad (55)$$

$$\dot{z}_{\theta 12}^a = -c_{\theta 12} c_3 z_3 f_2(z_2, z_3+\alpha_3) \quad (56)$$

$$\dot{z}_{\theta 2}^a = c_{\theta 2} z_3 f_3'(z_2, z_3+\alpha_3) \quad (57)$$

which results in

$$\dot{V}_\theta^a(\mathbf{z},\mathbf{z}_\theta) \le \frac{d_2k_2}{d_3k_3} z_1 f_1(z_1) - z_{2,3}^T Q^a(t)\, z_{2,3} \quad (58)$$

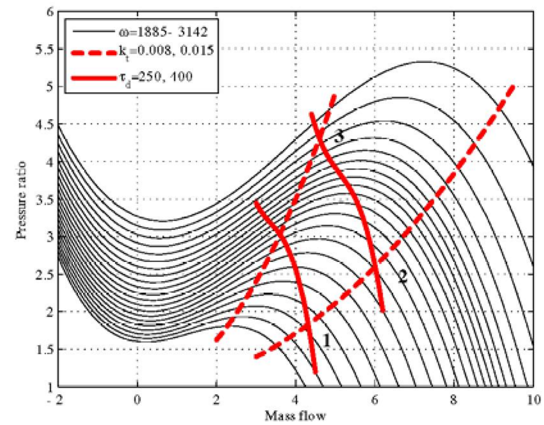

Fig. 3. Equilibrium points represented in compressor map

being equal to (51). Hence, by following the same arguments as for (51), it is concluded that all solutions $(\mathbf{z}(t), \mathbf{z}_\theta(t))$ are uniformly bounded and $\lim_{t\to\infty}\mathbf{z}\to\mathbf{0}$.

The overall system $(\mathbf{z},\mathbf{z}_\theta)$ is not strictly passive since $\dot{V}_\theta(\mathbf{z},\mathbf{z}_\theta)$ in (51) or (58) is only negative definite in $\mathbf{z}$. It is output strictly passive with respect to z_3, but it is not zero state observable since it can not be guaranteed that $\mathbf{z}_\theta$ converges to zero. Hence, the result of passive feedback interconnections can not be used to introduce the passive control part u_3 in the same manner as was done in the case of non-adaptive control law. owever, with the passive part (36) and the related assumptions, it is still possible to incorporate a passive control part. This can be seen by defining

$$V_{s,\theta}(\mathbf{z},\mathbf{z}_\theta,\mathbf{z}_{u3}) = V_\theta(\mathbf{z},\mathbf{z}_\theta) + V_{s,u3}(\mathbf{z}_{u3}) \quad (59)$$

and calculating the time derivative

$$\dot{V}_{\theta s}(\mathbf{z},\mathbf{z}_\theta,\mathbf{z}_{u3}) \le -W(\mathbf{z}) - W_{s,u3}(\mathbf{z}_{u3}) \quad (60)$$

where $W_{s,u3}(\mathbf{z}_{u3})$ and $W(\mathbf{z})$ are positive definite in their arguments, as can be seen from (36) and (51). Hence, by following the same arguments as for (51) and (58), it is concluded that $(\mathbf{z}(t), \mathbf{z}_\theta(t), \mathbf{z}_{u3})$ is uniformly bounded and $\lim_{t\to\infty}(\mathbf{z},\mathbf{z}_{u3})\to\mathbf{0}$.

Note that a passive, output strictly passive or input strictly passive Σ_{u3} will still give bounded solutions of overall system and convergence of $\mathbf{z}$ to the origin. owever, no conclusion can be drawn on the convergence of Σ_{u3} states. This can be seen by evaluating (60) when $W_{s,u3}(\mathbf{z}_{u3})$ is not positive definite in $\mathbf{z}_{u3}$, but rather positive semi definite.

VII. Simulation

The compressor map used for simulations is the same as used in [13]. This is a map based on measurement data, for which third order approximations in both compressor speed and mass flow is done to make the map continuos in these variables. Using (6) and (7), the throttle characteristics can be plotted in the compressor map. This is illustrated in Fig. 3 for two different throttle openings (represented by different k_t's), where the intersection of compressor and throttle characteristics constitutes the possible equilibrium of the system This means that freedom to choose a desired operating point for the compression system using only the drive torque as actuator is limited to some point on the throttle characteristics.

Using (7) and (8), the torque characteristic can be plotted in the compressor map. This is illustrated in Fig. 3 for two constant torque inputs of different amplitude. The system equilibrium is then given by the intersection of throttle and torque characteristic.

Simulations will go through a scenario in which the system is initially operating in point 1 of Fig. 3 ($k_t = 0.015$ and $\tau_d = 250$). The system is then driven to operating point 2 ($k_t = 0.015$ and $\tau_d = 400$) after 500 seconds, which involves a change of torque input for the uncontrolled system and a change of desired equilibrium for the controlled system. The system is driven to operating point 3 ($k_t = 0.008$ and $\tau_d = 400$) after 1500 seconds, which involves a change in the throttle opening for the both uncontrolled and controlled system in addition to a change of desired equilibrium for the controlled system. Change of throttle opening and equilibrium points are done with a step change of the parameter in question, in series with a first order filter ($T = 100$). This is done to get a realistic response from the control law with respect to commanded torque, since a step change introduces a relatively large amplitude of error variables. Without this filter the system still poses the same qualitative response as with the filter, but the commanded torque becomes unreasonably large in the transients.

For the scenario described, the throttle openings constitutes equilibrium points for which the compressor characteristics has a negative and positive slope in w. From the literature it is well known that a negative slope constitutes stable equilibriums. For positive slope however, the system need not be stable and surge can occur.

The open loop response is shown in Fig. 4. This simulation shows stable behavior for the first two operating points, before eventually entering surge in the last operating point. From the plot of compressor speed, it seems like this state is not oscillating. owever, a closer examination reveals relative small oscillation also for this state.

The closed loop response of the stabilizing control laws from section III is shown in Fig. 5, for control gains $c_3 = 20$, $c_4 = 1$ and $c_4 = 0.1$. For $c_4 = 1$ the response is practically identical for the various implementations, whereas for $c_4 = 0.1$ one can identify some differences. Furthermore, it can be seen that the amplitude of c_4 influences the convergence rate. This is especially the case for control law a.

The closed loop response of the adaptive control laws from section VI is shown in Fig. 6 and Fig. 7, for control gains $c_3 = 20$, $c_4 = 1$, $c_{\theta 1} = 10^{-5}$ and $c_{\theta 2} = 10^{-10}$. These simulations where conducted with the adaptive gain θ_1 being bounded to a positive value. The justification for this is found in the physics, when it is known that this parameter is positive. If this parameter was not bounded and the gain $c_{\theta 1}$ was chosen relatively high, the system states showed the same transients as in Fig. 6, but the control input would be unreasonably large. Choosing $c_{\theta 1}$ relatively low resulted in reasonable amplitude of control input, but the system will enter surge while "waiting" for θ_1 to converge (converge to some value that stabilizes the system). The various control laws seems to give the same response for system states and input also in this case (note different scale for τ_d relative to previous plots). Furthermore, the response for θ_2 is almost identical for the different control laws. Some differences are found in the θ_1 response, where it is also evident that the lower saturation of parameter estimate have been active. The actual values of the two unknown parameters are $k_{\theta 1} = 0.0497$ and $k_{\theta 2} = 0.0285$. These simulations show that θ_1 does not converge to its actual value, whereas θ_2 does.

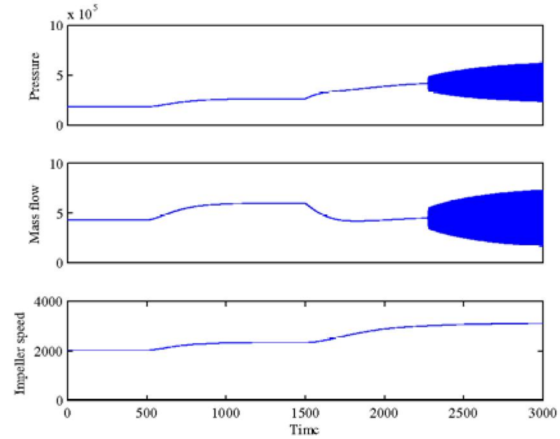

Fig. 4. Open loop simulation

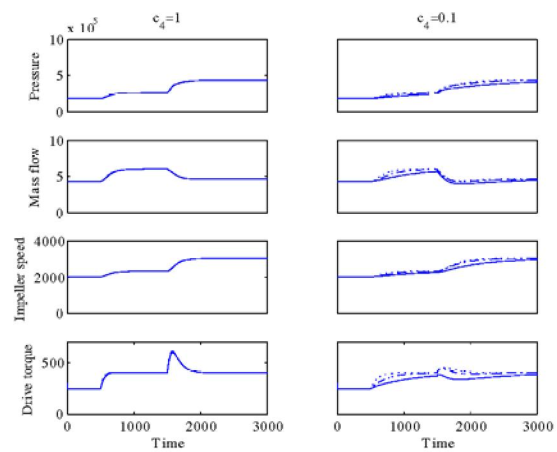

Fig. 5. System states and control input of closed loop simulation using stabilizing control laws from section III. Control laws a, b, and c are represented by solid, dash-dotted and dotted lines respectively.

VIII. Concluding Remarks

The various control laws can be divided into a stabilizing part, a robust part, a passive part and an adaptive part. It is shown that the overall control law can consist of all these

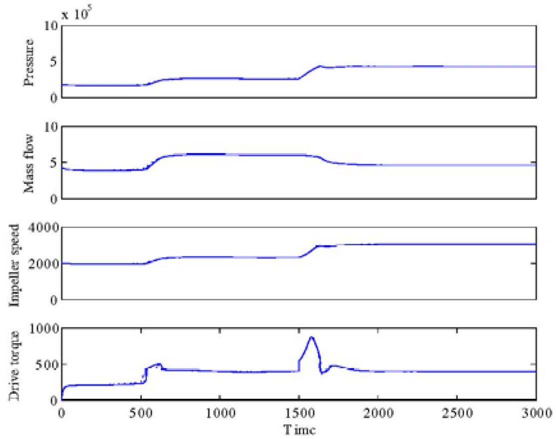

Fig. 6. System states and control input of closed loop simulation using stabilizing control laws from section VI. Control laws a, b, and c are represented by solid, dash-dotted and dotted lines respectively.

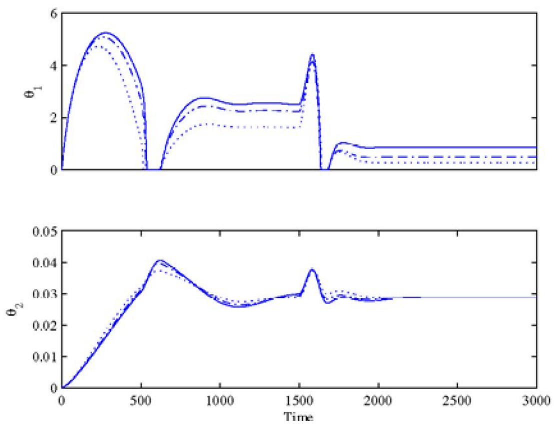

Fig. 7. Adaptive gains of closed loop simulation using stabilizing control laws from section VI. Control laws a, b, and c are represented by solid, dash-dotted and dotted lines respectively.

parts or a selection, but the stabilizing part must always be present to guarantee stability. Furthermore, the stabilizing part can be implemented in three different ways, depending on how the controller compensates for compressor torque. Simulation using only the stabilizing part and simulation using the stabilizing and adaptive part showed small differences for the various implementations of the stabilizing part.

The control laws makes use of pressure downstream the compressor. This can either be implemented using a model (compressor map) or measurement. If implemented using a compressor map, this introduces uncertainty due to model based cancellation (addressed in section V). This is especially the case to the left of the surge line, when limited knowledge of steady state compressor behavior is available in this region. Furthermore, the uncertainty involved in the compressor map gives rise to uncertainty of equilibrium points for a practical installation, as they are identified using this map. A solution to this problem might be to estimate the equilibrium points.

An alternative to estimate k_{θ_1} is to use identification techniques, such as reported in [14]. These techniques can also be used to get a good initial estimate for the variables, when an adaptive scheme is preferred.

All control laws require measurement of mass flow. owever, measurement of mass flow for dynamic purposes is troublesome. Hence, a mass flow observer should be incorporated in the control law before it can be implemented in practice.

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