# Attitude Stabilization of an Underactuated Rigid Spacecraft 

Siv.Ing Thesis<br>Spring 2003

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## Preface

This work is the concluding thesis of the Siv.ing. education at the Norwegian University of Technology and Science(NTNU). It has been carried out at the Department of Engineering Cybernetics(ITK). I would like to thank my supervisor Professor Kristin Y. Pettersen and advisor Assistant Professor Jan Tommy Gravdahl for their support,valuable advice and interesting discussions during this work.

Most of all I would like to thank my fiancée, Maria, for her love and support during this work. I would also like to thank all my friends and fellow students for all the fun and hard work during five wonderful years in Trondheim.

Kjell Magne Fauske
Trondheim 2003-06-10

## Summary

The topic of this thesis is the feedback stabilization of the attitude of an underactuated rigid spacecraft. Underactuated mechanical systems are characterized by the fact that there are more degrees of freedom than actuators. In the case of the rigid spacecraft, we try to stabilize the three-axis attitude with only two available actuators. The problem is interesting both from a practical and theoretic point of view and has received much attention in the last decades.

The spacecraft is modelled as an ideal rigid body. To represent the spacecraft's attitude the $(\mathrm{w}, z)$-parameterization is used. The $(\mathrm{w}, z)$-parameterization is a relatively new parameterization and has properties that makes it very interesting for the attitude control problem. It is a minimal and compact parameterization with the singularity moved as far away from the origin as possible, and the motion of the $z$-axis is decoupled from the rest of the system.

It is shown that the underactuated rigid spacecraft model does not satisfy Brockett's necessary condition, i.e, it can not be stabilized by a continuous time-invariant state feedback. However, it is possible to achieve stabilization about an equilibrium manifold. It is shown that when using the ( $\mathrm{w}, z$ ) parameterization it is very simple to find such controllers.

The purpose of this thesis is to apply the results of Mazenc et al. (2002) to solve the open problem of determining continuous controllers that globally stabilize the attitude of the underactuated spacecraft. It is demonstrated how Mazenc et al. (2002) solves the open problem of determining explicit time-varying, periodic smooth feedbacks that globally uniformly asymptotically stabilize an underactuated surface vessel. Unfortunately the spacecraft model has no damping and the kinematics are more complicated. Direct application is therefore not possible. However, Mazenc et al. (2002) provides several useful tools and methods that can be used.

An attempt is made to solve the attitude stabilization problem by first solving the subproblems of spin-axis stabilization and angular velocity stabilization. By using tools from Mazenc et al. (2002) a smooth time-varying periodic controller is derived that globally uniformly asymptotically stabilize the angular velocities of an underactuated spacecraft. By simulations it is indicated that the two controllers can be combined to achieve partial attitude stabilization. Unfortunately no proof is available at the time of writing.

The problem of determining globally stabilizing control laws is still open. However it is probable that such control laws can be found using the tools in Mazenc et al. (2002).

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## Chapter 1

## Introduction

The topic of this thesis is attitude stabilization of an underactuated rigid spacecraft. Underactuated mechanical systems are characterized by the fact that there are more degrees of freedom than actuators. In the case of the rigid spacecraft, we try to stabilize the three-axis attitude with only two available actuators.

The motivation for studying this problem is both practical and theoretical. Usually an actuator failure is handled by incorporating redundancy in the design. The disadvantage of this approach is higher weight and a more complicated mechanical system. An alternative is to use more complicated controllers that manage to achieve the control objectives with only two actuators. From a theoretical point of view the stabilization of an underactuated system is a challenging problem. Many interesting control-theoretic questions have to be answered and the control problem is highly nonlinear.

In recent years there has been an increasing interest in space-related activities in Norway, both from the industry and educational institutes. Hopefully this thesis will contribute to increase the knowledge and interest in attitude control systems for spacecrafts.

### 1.1 Space-related activities in Norway

Norway has a long tradition of space-related research. The main emphasis in space-based activities are currently on ESA projects and on sounding rockets from Andøya Rocket Range. Norwegian universities and industry have contributed to many space projects, however no Norwegian built satellite has been launched into space. Hopefully this will change in the near future. Two satellites are at the time of writing being planned, and the author has been fortunate to have participated in the study and specification phase for both satellites

NSAT-1 The NSAT-1 mission, initiated by the Norwegian Defence Research Establishment (FFI), is a mission for demonstration of localization of maritime vessels by passive detection of their X-band navigation radar, and subsequent direction finding and determination of the geographic position (Narheim et al., 2001). The intention is to use the satellite to monitor the maritime activities in Norwegian ocean areas. A high performance micro-satellite is required for this concept.

NCUBE The Norwegian Student Satellite Project aims to design, build, integrate, test and launch a small satellite. The project is primarily a collaboration between students at the Norwe-
gian University of Technology and Science (NTNU), the University College of Narvik (HiN), and the Agricultural University of Norway (NLH). Andøya Rocketrange (ARS) and the Norwegian Space Centre (NRS) provides project management and financial support.

The satellite concept Cubesat, developed by Stanford University, has been chosen as a framework for NCUBE. The satellite will be $10 \times 10 \times 10 \mathrm{~cm}$, weighing a maximum of 1 kg . The satellite will be stabilized by a gravity gradient boom and three electromagnetic coils (Fauske, 2002) ${ }^{1}$.

### 1.2 Nonholonomic systems

Control of underactuated mechanics systems has been a very active topic of research during the last decades. The research on underactuated systems is a continuation on the research on nonholonomic systems, as many underactuated systems are subject to nonholonomic constraints. Nonholonomic constraints can be divided into first-order and second-order nonholonomic constraints. First-order constraints are non-integrable constrains on the form $\Phi(q, \dot{q})=0$, where $q$ and $\dot{q}$ are generalized coordinates and velocities. Second-order constraints are on the form $\Phi(q, \dot{q}, \ddot{q})=0$, and constrain the acceleration of the system. A common property of nonholonomic systems is that they can not be stabilized time-invariant pure state feedback. An excellent introduction to nonholonomic systems can be found in Kolmanovsky and McClamroch (1995). The number of publications on nonholonomic and underactuated systems is extensive. In the next section some of the most interesting publications concerning underactuated spacecrafts are presented.

### 1.3 Previous work

There exist numerous research articles on the problem of attitude stabilization of spacecrafts. Most of these deals with the case of complete control actuation using either reaction wheels, thrusters or magnetic actuators. Some contributions from Scandinavian scientist are for instance Egeland and Godhavn (1994), Dalsmo and Egeland (1997), Skullestad and Gilbert (2000) and Wiśniewski and Blanke (1999).

The angular velocity control of a rigid body with only one or two controls has been studied extensively in the literature. The issue of feedback stabilization of the angular velocities has been solved using various approaches. In Brockett (1985) it was shown by finding a Lyapunov function, that the null solution of the angular velocity equations is asymptotically stabilizable by two control torques aligned with two principal axes if the uncontrolled axis is not an axis of symmetry. The angular velocity of a rigid body can in fact be asymptotically stabilized by smooth feedback with a single control as long as the control is not aligned with a principal axis (Aeyels and Szafranski, 1988).

Stabilization of the angular velocities of a symmetric rigid body is addressed in Andriano (1993) and Outbib (1994). It is shown that that the angular velocities can be globally stabilized by means of linear feedback when two control torques act on the body. In Reyhanoglu (1996) it is shown that the angular velocity equation of a rigid body with two control torques cannot be exponentially stabilized using a $\mathcal{C}^{1}$ feedback. Discontinuous feedback laws are proposed that achieve asymptotic stability with exponential convergence rate.

[^0]In Mazenc and Astolfi (2000) the problem of semi-global stabilization of the angular velocity of an underactuated rigid body in the presence of model errors is addressed and solved using a smooth, time-varying, dynamic, output feedback control law. Robustness is also addressed in Morin (1996), where homogeneity properties of the system are exploited, and in Astolfi and Rapaport (1997). For more references on stabilization of the angular velocity of a rigid body refer to Tsiotras and Doumtchenko (2000) and references therein.

The more difficult problem of feedback stabilization of both the the angular velocities and attitude equations has also received much attention. One of the earliest investigations of the attitude control problem was done in Crouch (1984), where necessary and sufficient conditions for the controllability of a rigid body in the case of one, two and three independent control torques was provided. In the case of momentum exchange devices it was shown that controllability is impossible with fewer than three devices.

In Byrnes and Isidori (1991) the longstanding problem concerning the existence of a timeinvariant smooth state feedback locally asymptotically stabilizing an underactuated rigid spacecraft was settled in the negative. However stabilization about an attractor is possible, inducing a closed-loop system with trajectories tending to a revolute motion about a principal axis. A discontinuous control strategy was suggested in Krishnan et al. (1994). By switching between various controllers a sequence of maneuvers were performed that stabilized the spacecraft to any equilibrium attitude in finite time.

An article by Samson (1991) triggered the discovery that many systems that can not be stabilized by continuous state-feedback can in fact be stabilized by smooth time-varying feedback. A locally stabilizing smooth time-varying feedback was derived in Morin et al. (1995) by using center manifold theory combined with averaging and Lyapunov techniques. However, due to the smoothness of the control laws, the rate of convergence is only polynomial in the worst case. Similar results where derived in Coron and Keraï (1996) for the general case of torques that are not aligned with the principal axes of the satellite.

A stronger result was achieved in Morin and Samson (1997) where the attitude of the underactuated rigid spacecraft was locally, exponential stabilized with respect to a given dilation. The controller was periodic, time varying and non-differentiable at the origin and the construction relied on the properties of homogeneous systems.

For the axi-symmetric rigid body there exist a wide range of results. However, stabilization is only possible for the restricted case of zero spin rate about the unactuated axis. Spinstabilization with two control torques is addressed in Tsiotras and Longuski (1994) based on a new formulation for the attitude dynamics. The new attitude formulation, described in Tsiotras and Longuski (1995), was subsequently used in Tsiotras et al. (1995) and Tsiotras and Luo (1996) to derive a time-invariant discontinuous control law that achieves arbitrary reorientation of the spacecraft. Due to the properties of the new parameterization, the control laws derived were especially simple and elegant. In a more recent paper Tsiotras and Luo (2000), saturated tracking and stabilization laws were developed for the underacutated axi-symmetric rigid body under the assumption of zero spin rate. Another time-varying tracking law was developed in Behal et al. (2002) using a Lyapunov-based approach and by exploiting several characteristics of the quaternion attitude formulation. The spin rate was not required to be zero, however the spacecraft could only be driven to an arbitrarily small neighborhood of the origin.

The topic of feasible trajectory generation for the underactuated spacecraft has not received much attention. An exception is Tsiotras and Luo (2000) where feasible trajectories are generated for the axi-symmetric rigid body using the notion of differential flatness. For an ex-
cellent overview of developments in control of the underactuated spacecraft see Tsiotras and Doumtchenko (2000) and references therein.

### 1.4 Contributions of this thesis

- An extensive list of references on the subject of control of underactuated spacecrafts have been compiled. This will serve as an excelent starting point for further study.
- The relatively unknown $\left(\mathrm{w}_{z}\right)$-attitude paramaterization is presented as an usefull tool for the attitude stabilization problem.
- The most important properties of an underactuated spacecraft are presented and compared with other underactuated systems.
- A time-varying periodic controller is proposed for the angular velocity stabilization problem. The controller achieves global uniform asymptotic stability. Strict Lyapunov functions are constructed to prove stability.
- The locally exponential stabilizing controller in Morin and Samson (1997) is extended to the $(\mathrm{w}, z)$-parameterization.


### 1.5 Outline of the thesis

The thesis is organized as follows:

- Chapter 2: Different parameterizations of the attitude and their properties are described, with an emphasis on the relatively new ( $\mathrm{w}, z$ )-parameterization.
- Chapter 3: An introduction to spacecraft dynamics, actuators and space environment is given.
- Chapter 4: A complete model of an underactuated spacecraft is presented and some important control properties are discussed.
- Chapter 5: The results in Mazenc et al. (2002) concerning the global uniform asymptotic stabilization of an underactuated surface vessel are presented along with a discussion of how the results can be extended to the underactuated spacecraft stabilization problem.
- Chapter 6: Feedback laws are presented that globally stabilizes the spin axis and angular velocities. An attempt is made to solve the partial and complete attitude stabilization problem. Feedback laws are presented that extends the results of Morin and Samson (1997) to the $(\mathrm{w}, z)$-parameterization.
- Chapter 7: Conclusions and recommendations for further work are given.
- Appendix A: Some background theory is presented.
- Appendix B: The Newton-Euler equations for rigid bodies are derived.


## Chapter 2

## Attitude parameterizations

Euler's equations of motion are commonly used to describe the dynamics of a rigid spacecraft. The equations of motion provide a complete and well-defined framework. For the kinematics the situation is different, due to the fact that the rotation matrix, which describes the relative orientation between two reference frames, can be parameterized in more than one way. Which parametrization to use is clearly problem dependent. This chapter discusses different attitude parameterizations and describes in detail the relatively new $(\mathrm{w}, z)$-parametrization.

### 2.1 The rotation matrix

The rotation matrix, also called the direction cosine matrix, has three interpretations:

- Describes the mutual orientation between two coordinate frames, where the column vector are cosines of the angles between the two frames.
- Transforms vectors represented in one reference frame to another.
- Rotates a vector within a reference frame.

The rotation matrix $\mathbf{R}$ from frame $a$ to $b$ is denoted $\mathbf{R}_{b}^{a}$ and is an element in $S O(3)$, which is defined as

$$
\begin{equation*}
S O(3)=\left\{\mathbf{R} \mid \mathbf{R} \in \mathbb{R}^{3 \times 3}, \mathbf{R}^{\mathrm{T}} \mathbf{R}=\mathbf{I} \text { and } \operatorname{det} \mathbf{R}=1\right\} \tag{2.1}
\end{equation*}
$$

where $\mathbf{I}$ is the $3 \times 3$ identity matrix.
From the orthogonality property, $\mathbf{R}^{\mathrm{T}} \mathbf{R}=\mathbf{I}$, it can be shown that the time derivate of the rotation matrix is (Egeland and Gravdahl, 2002)

$$
\begin{equation*}
\dot{\mathbf{R}}_{b}^{a}=\boldsymbol{\omega}_{a b}^{a} \times \mathbf{R}_{b}^{a}, \tag{2.2}
\end{equation*}
$$

where $\boldsymbol{\omega}_{a b}^{a}$ is the angular velocity of frame $b$ relative to frame $a$ represented in the $a$ frame. The angular velocity has the property $\boldsymbol{\omega}_{a b}^{a}=-\boldsymbol{\omega}_{b a}^{a}$, giving

$$
\begin{equation*}
\dot{\mathbf{R}}_{b}^{a}=-\boldsymbol{\omega}_{b a}^{a} \times \mathbf{R}_{b}^{a}=\mathbf{R}_{b}^{a} \times \boldsymbol{\omega}_{b a}^{a} . \tag{2.3}
\end{equation*}
$$

The cross product can be rewritten using the skew-symmetric cross product operator $\mathbf{S}(\boldsymbol{\omega})$ :

$$
\boldsymbol{\omega} \times=\mathbf{S}(\boldsymbol{\omega})=\left[\begin{array}{ccc}
0 & -\omega_{z} & \omega_{y}  \tag{2.4}\\
\omega_{z} & 0 & -\omega_{x} \\
-\omega_{y} & \omega_{x} & 0
\end{array}\right], \quad \boldsymbol{\omega}=\left[\begin{array}{c}
\omega_{x} \\
\omega_{y} \\
\omega_{z}
\end{array}\right]
$$

Using (2.4) we can rewrite (2.2) as

$$
\begin{equation*}
\dot{\mathbf{R}}_{b}^{a}=\mathbf{S}\left(\boldsymbol{\omega}_{a b}^{a}\right) \mathbf{R}_{b}^{a}=\mathbf{R}_{b}^{a} \mathbf{S}\left(\boldsymbol{\omega}_{a b}^{b}\right) \tag{2.5}
\end{equation*}
$$

### 2.2 Euler parameters

The Euler parameters, also called unit quaternions, are attractive due to their nonsingular parametrization and linear kinematic differential equations if the angular velocities are known. The quaternion representation requires much less computations than for instance the Euler angles representation, and is therefore useful in applications where computer resources are limited.

The Euler parameters are defined in terms of the principal rotation angle $\theta$ and the principal line components $k_{i}$ as follows:

$$
\begin{equation*}
\eta=\cos \frac{\theta}{2}, \quad \varepsilon_{i}=k_{i} \sin \frac{\theta}{2} \quad i=1,2,3 \tag{2.6}
\end{equation*}
$$

They also satisfy the constraint:

$$
\begin{equation*}
\eta^{2}+\varepsilon_{1}^{2}+\varepsilon_{2}^{2}+\varepsilon_{3}^{2}=1 \tag{2.7}
\end{equation*}
$$

The Euler parameters kinematic differential equations are given as:

$$
\left[\begin{array}{l}
\dot{\eta}  \tag{2.8}\\
\dot{\varepsilon}
\end{array}\right]=\frac{1}{2}\left[\begin{array}{ccc}
-\varepsilon_{1} & -\varepsilon_{2} & -\varepsilon_{3} \\
\eta & -\varepsilon_{3} & \varepsilon_{2} \\
\varepsilon_{3} & \eta & -\varepsilon_{1} \\
-\varepsilon_{2} & \varepsilon_{1} & \eta
\end{array}\right]\left[\begin{array}{l}
\omega_{1} \\
\omega_{2} \\
\omega_{3}
\end{array}\right]
$$

In component form:

$$
\begin{align*}
\dot{\eta} & =-\frac{1}{2}\left(\varepsilon_{1} \omega_{1}+\varepsilon_{2} \omega_{2}+\varepsilon_{3} \omega_{3}\right)  \tag{2.9a}\\
\dot{\varepsilon}_{1} & =\frac{1}{2}\left(\eta \omega_{1}-\varepsilon_{3} \omega_{2}+\varepsilon_{2} \omega_{3}\right)  \tag{2.9b}\\
\dot{\varepsilon}_{2} & =\frac{1}{2}\left(\varepsilon_{3} \omega_{1}+\eta \omega_{2}-\varepsilon_{1} \omega_{3}\right)  \tag{2.9c}\\
\dot{\varepsilon}_{3} & =\frac{1}{2}\left(-\varepsilon_{2} \omega_{1}+\varepsilon_{1} \omega_{2}+\eta \omega_{3}\right) \tag{2.9d}
\end{align*}
$$

### 2.3 Rodrigues parameters

The classical and modified parameters can be interpreted as the coordinates resulting from a stereographic projection of the four-dimensional Euler parameter hypersphere onto a threedimensional hyperplane (Schaub et al., 1995). The difference between them is how the projection point and mapping hyperplane is chosen.

### 2.3.1 The classical Rodrigues parameters

The classical Rodrigues parameters can be derived from the Euler parameters with the transformation

$$
\begin{equation*}
q_{i}=\frac{\varepsilon_{i}}{\eta}, \quad i=1,2,3 . \tag{2.10}
\end{equation*}
$$

Combining (2.10) and (2.6) yields

$$
\begin{equation*}
q_{i}=k_{i} \tan \frac{\theta}{2} . \tag{2.11}
\end{equation*}
$$

Clearly the the classical Rodrigues parameters has singular condition for $\theta= \pm \pi$, where $q_{i} \rightarrow \infty$. The kinematic differential equations can be derived from (2.8), giving:

$$
\left[\begin{array}{l}
\dot{q}_{1}  \tag{2.12}\\
\dot{q}_{2} \\
\dot{q}_{3}
\end{array}\right]=\frac{1}{2}\left[\begin{array}{ccc}
1+q_{1}^{2} & q_{1} q_{2}-q_{3} & q_{1} q_{3}+q_{2} \\
q_{1} q_{2}+q_{3} & 1+q_{2}^{2} & q_{2} q_{3}-q_{1} \\
q_{3} q_{1}-q_{2} & q_{3} q_{2}+q_{1} & 1+q_{3}^{2}
\end{array}\right]\left[\begin{array}{l}
\omega_{1} \\
\omega_{2} \\
\omega_{3}
\end{array}\right]
$$

In component form it is:

$$
\begin{align*}
& \dot{q}_{1}=\frac{1}{2}\left(\left(1+q_{1}^{2}\right) \omega_{1}+\left(q_{1} q_{2}-q_{3}\right) \omega_{2}+\left(q_{1} q_{3}+q_{2}\right) \omega_{3}\right)  \tag{2.13a}\\
& \dot{q}_{2}=\frac{1}{2}\left(\left(q_{1} q_{2}+q_{3}\right) \omega_{1}+\left(1+q_{2}^{2}\right) \omega_{2}+\left(q_{2} q_{3}-q_{1}\right) \omega_{3}\right)  \tag{2.13b}\\
& \dot{q}_{3}=\frac{1}{2}\left(\left(q_{3} q_{1}-q_{2}\right) \omega_{1}+\left(q_{3} q_{2}+q_{1}\right) \omega_{2}+\left(1+q_{3}^{2}\right) \omega_{3}\right) \tag{2.13c}
\end{align*}
$$

### 2.3.2 The modified Rodrigues parameters

The modified Rodrigues parameters can be derived from the Euler parameters with the transformation

$$
\begin{equation*}
\sigma_{i}=\frac{\varepsilon_{i}}{1+\eta}, \quad i=1,2,3 \tag{2.14}
\end{equation*}
$$

Combining (2.10) and (2.14) yields

$$
\begin{equation*}
\sigma_{i}=k_{i} \tan \frac{\theta}{4} . \tag{2.15}
\end{equation*}
$$

Clearly the the modified Rodrigues parameters has a singular condition for $\theta= \pm 2 \pi$, which means that the nonsingular rotation range is two times larger than the nonsingular rotation range for the classical Rodrigues parameters. Insertion of (2.14) in (2.8) yields:

$$
\left[\begin{array}{l}
\dot{\sigma}_{1}  \tag{2.16}\\
\dot{\sigma}_{2} \\
\dot{\sigma}_{3}
\end{array}\right]=\frac{1}{2}\left[\begin{array}{ccc}
1+\sigma_{1}^{2}-\sigma_{2}^{2}-\sigma_{3}^{2} & 2\left(\sigma_{1} \sigma_{3}+\sigma_{2}\right) & 2\left(\sigma_{1} \sigma_{2}-\sigma_{3}\right) \\
2\left(\sigma_{2} \sigma_{1}+\sigma_{3}\right) & 1-\sigma_{1}^{2}+\sigma_{2}^{2}-\sigma_{3}^{2} & 2\left(\sigma_{2} \sigma_{3}-\sigma_{1}\right) \\
2\left(\sigma_{3} \sigma_{1}-\sigma_{2}\right) & 2\left(\sigma_{3} \sigma_{2}+\sigma_{3}\right) & 1-\sigma_{1}^{2}-\sigma_{2}^{2}+\sigma_{3}^{2}
\end{array}\right]\left[\begin{array}{l}
\omega_{1} \\
\omega_{2} \\
\omega_{3}
\end{array}\right]
$$

In component form it is:

$$
\begin{align*}
& \dot{\sigma}_{1}=\frac{1}{2}\left(1+\sigma_{1}^{2}-\sigma_{2}^{2}-\sigma_{3}^{2}\right) \omega_{1}+\left(\sigma_{1} \sigma_{3}+\sigma_{2}\right) \omega_{2}+\left(\sigma_{1} \sigma_{2}-\sigma_{3}\right) \omega_{3}  \tag{2.17a}\\
& \dot{\sigma}_{2}=\left(\sigma_{2} \sigma_{1}+\sigma_{3}\right) \omega_{1}+\frac{1}{2}\left(1-\sigma_{1}^{2}+\sigma_{2}^{2}-\sigma_{3}^{2}\right) \omega_{2}+\left(\sigma_{2} \sigma_{3}-\sigma_{1}\right) \omega_{3}  \tag{2.17b}\\
& \dot{\sigma}_{3}=\left(\sigma_{3} \sigma_{1}-\sigma_{2}\right) \omega_{1}+\left(\sigma_{3} \sigma_{2}+\sigma_{3}\right) \omega_{2}+\frac{1}{2}\left(1-\sigma_{1}^{2}-\sigma_{2}^{2}+\sigma_{3}^{2}\right) \omega_{3} \tag{2.17c}
\end{align*}
$$


(a)

(b)

Figure 2.1: Rotation sequence of the ( $\mathrm{w}, z$ )-parameterization. Left hand side shows the initial rotation about the body $z$-axis. Right hand side shows the resulting coordinate system after the second rotation.

### 2.4 The ( $\mathrm{w}, z$ ) parametrization

The ( $\mathrm{w}, z$ ) parametrization is a relatively new approach to describe the kinematics of a rotating rigid body. The formulation describes the relative orientation of two reference frames using two perpendicular rotations, thus complementing the Eulerian angles (three rotations) and the Euler parameters (one rotation) (Tsiotras and Longuski, 1995). The parametrization is probably unfamiliar to the reader and therefore it is derived in detail. The material in this section is based on Tsiotras and Longuski $(1995,1996)$.

Consider a reference frame $i$, defined by three orthogonal unit vectors $\vec{i}_{1}, \vec{i}_{2}$ and $\vec{i}_{3}$. Another reference frame $b$ is fixed to the body and is defined by the three orthogonal unit vectors $\vec{b}_{1}, \vec{b}_{2}$ and $\vec{b}_{3}$. The rotation matrix $\mathbf{R}$ from the reference frame to the body frame is decomposed using two successive rotations:

$$
\begin{equation*}
\mathbf{R}(\mathrm{w}, z)=\mathbf{R}_{b}^{i}=\mathbf{R}_{2}(\mathrm{w}) \mathbf{R}_{1}(z) \tag{2.18}
\end{equation*}
$$

The two rotations are illustrated in Figure 2.1. The first rotation represents a simple rotation about the body $z$-axis, and is given by

$$
\mathbf{R}_{1}(z)=\left[\begin{array}{ccc}
\cos z & \sin z & 0  \tag{2.19}\\
-\sin z & \cos z & 0 \\
0 & 0 & 1
\end{array}\right]
$$

The intermediate reference frame is denoted $o$, defined by the three orthogonal unit vectors $\vec{o}_{1}, \vec{o}_{2}$ and $\vec{o}_{3}$.

The next step is to find an expression for the second rotation matrix $\mathbf{R}_{2}(\mathrm{w})$. Recall that a rotation matrix can be computed using an angle-axis description, which corresponds to a rotation by an angle $\theta$ about an unit vector $\vec{u}$ (Egeland and Gravdahl, 2002):

$$
\begin{equation*}
\mathbf{R}=\mathbf{I}+\sin (\theta) \mathbf{S}(\vec{u})+(1-\cos \theta) \mathbf{S}^{2}(\vec{u}) \tag{2.20}
\end{equation*}
$$

Consider the vector $\mathbf{v}^{o}=[0,0,1]^{\mathrm{T}}$, which is along the $z_{o}$-axis. Transforming it to the $b$ frame gives

$$
\left[\begin{array}{l}
a  \tag{2.21}\\
b \\
c
\end{array}\right]=\mathbf{R}_{2}(\mathrm{w})\left[\begin{array}{l}
0 \\
0 \\
1
\end{array}\right],
$$

where $[a, b, c]^{\mathrm{T}}$ is the third column in the rotation matrix. The axis $\vec{o}_{3}$ can therefore be written as

$$
\begin{equation*}
\vec{o}_{3}=a \vec{b}_{1}+b \vec{b}_{2}+c \vec{b}_{3} . \tag{2.22}
\end{equation*}
$$

By symmetry the relation

$$
\begin{equation*}
\vec{b}_{3}=-a \vec{o}_{1}-b \vec{o}_{2}+c \vec{o}_{3} \tag{2.23}
\end{equation*}
$$

also holds. The angle between $\vec{b}_{3}$ and $\vec{o}_{3}$ is simply

$$
\begin{equation*}
\theta=\cos ^{-1}\left(\vec{o}_{3} \cdot \vec{b}_{3}\right)=\cos ^{-1} c . \tag{2.24}
\end{equation*}
$$

Now the axis of rotation can be found with

$$
\begin{equation*}
\vec{u}=\frac{\overrightarrow{o_{3}} \times \vec{b}_{3}}{\left\|\overrightarrow{i_{3}} \times \vec{b}_{3}\right\|} \tag{2.25}
\end{equation*}
$$

Using (2.22) and (2.23) the axis of rotation can be written as

$$
\begin{equation*}
\vec{u}=\frac{b \vec{o}_{1}-c \vec{o}_{2}}{\sqrt{a^{2}+b^{2}}} . \tag{2.26}
\end{equation*}
$$

Now the angle-axis description of the rotation matrix can be used. Insertion of (2.26) and (2.24) into (2.20) gives

$$
\mathbf{R}_{2}(\mathrm{w})=\left[\begin{array}{ccc}
c+\frac{b^{2}}{1+c} & -\frac{a b}{1+c} & a  \tag{2.27}\\
-\frac{a b}{1+c} & c+\frac{a^{2}}{1+c} & b \\
-a & -b & c
\end{array}\right] .
$$

Expanding (2.18) gives the complicated matrix

$$
\mathbf{R}(\mathrm{w}, z)=\left[\begin{array}{ccc}
\frac{c \cos z+a b \sin z+\left(b^{2}+c^{2}\right) \cos z}{c \sin z+c} & \frac{c \sin z-a b \cos z+\left(b^{2}+c^{2}\right) \sin z}{1+c} & a  \tag{2.28}\\
-\frac{c \sin z+\left(c^{2}+a^{2}\right) \sin z+a b \cos }{1+c} z & \frac{c \cos z+\left(c^{2}+a^{2}+c \cos z-a b \sin z\right.}{1+c} & b \\
-b \sin z-a \cos z & -b \cos z-a \sin z & c
\end{array}\right]
$$

The representation in (2.18) is redundant since

$$
\begin{equation*}
a^{2}+b^{2}+c^{2}=1, \tag{2.29}
\end{equation*}
$$

hence one parameter can be eliminated. An elegant way of doing this is to use a stereographic projection of the point $(a, b, c)$ on the unit sphere on to the equatorial plane of the sphere. The stereographic projection is shown in Figure 2.2 and is defined as

$$
\begin{equation*}
\mathrm{w}:=\mathrm{w}_{1}+i \mathrm{w}_{2}=\frac{b-i a}{1+c}, \tag{2.30}
\end{equation*}
$$

where $i=\sqrt{-1}$. With some algebraic manipulation the the inverse relation can be found to be

$$
\begin{equation*}
a=\frac{i(\mathrm{w}-\overline{\mathrm{w}})}{1+|\mathrm{w}|^{2}}, \quad b=\frac{\mathrm{w}+\overline{\mathrm{w}}}{1+|\mathrm{w}|^{2}}, \quad c=\frac{1-|\mathrm{w}|^{2}}{1+|\mathrm{w}|^{2}}, \tag{2.31}
\end{equation*}
$$

where $\overline{\mathrm{w}}$ is the complex conjugate of w and $|\mathrm{w}|^{2}=\mathrm{w} \overline{\mathrm{w}}=\mathrm{w}_{1}^{2}+\mathrm{w}_{2}^{2}$. The basis of the projection is the point $(0,0,-1)$, which is the south pole $S$ of the unit sphere. Note that when $c=-1$ the projection has a singularity and $\mathrm{w} \rightarrow \infty$. The singularity corresponds to an upside-down orientation of the body. Using (2.31) with (2.27) gives the matrix


Figure 2.2: Stereographic projection of a point $(a, b, c)$ on a unit sphere on to a complex plane.

$$
\mathbf{R}_{2}(\mathrm{w})=\frac{1}{1+\mathrm{w}_{1}^{2}+\mathrm{w}_{2}^{2}}\left[\begin{array}{ccc}
1+\mathrm{w}_{1}^{2}-\mathrm{w}_{2}^{2} & 2 \mathrm{w}_{1} \mathrm{w}_{2} & -2 \mathrm{w}_{2}  \tag{2.32}\\
2 \mathrm{w}_{1} \mathrm{w}_{2} & 1-\mathrm{w}_{1}^{2}+\mathrm{w}_{2}^{2} & 2 \mathrm{w}_{1} \\
2 \mathrm{w}_{2} & -2 \mathrm{w}_{1} & 1-\mathrm{w}_{1}^{2}-\mathrm{w}_{2}^{2}
\end{array}\right]
$$

The matrix can be written more compactly using complex notation:

$$
\mathbf{R}_{2}(\mathrm{w})=\frac{1}{1+|\mathrm{w}|^{2}}\left[\begin{array}{ccc}
1+\operatorname{Re}\left(\mathrm{w}^{2}\right) & \operatorname{Im}\left(\mathrm{w}^{2}\right) & -2 \operatorname{Im}(\mathrm{w})  \tag{2.33}\\
\operatorname{Im}\left(\mathrm{w}^{2}\right) & 1-\operatorname{Re}\left(\mathrm{w}^{2}\right) & 2 \operatorname{Re}(\mathrm{w}) \\
2 \operatorname{Im}(\mathrm{w}) & -2 \operatorname{Re}(\mathrm{w}) & 1-|\mathrm{w}|^{2}
\end{array}\right]
$$

The total rotation matrix $\mathbf{R}$ using the $(\mathrm{w}, z)$ parametrization is

$$
\mathbf{R}=\frac{1}{1+\mathrm{w}_{1}^{2}+\mathrm{w}_{2}^{2}}\left[\begin{array}{ccc}
\left(1+\mathrm{w}_{1}^{2}-\mathrm{w}_{2}^{2}\right) \mathrm{c} z-2 \mathrm{w}_{1} \mathrm{w}_{2} \mathrm{~s} z & \left(1+\mathrm{w}_{1}^{2}-\mathrm{w}_{2}^{2}\right) \mathrm{s} z+2 \mathrm{w}_{1} \mathrm{w}_{2} \mathrm{c} z & -2 \mathrm{w}_{2}  \tag{2.34}\\
2 \mathrm{w}_{1} \mathrm{w}_{2} \mathrm{c} z-\left(1-\mathrm{w}_{1}^{2}+\mathrm{w}_{2}^{2}\right) \mathrm{s} z & 2 \mathrm{w}_{1} \mathrm{w}_{2} \mathrm{~s} z+\left(1-\mathrm{w}_{1}^{2}+\mathrm{w}_{2}^{2}\right) \mathrm{c} z & 2 \mathrm{w}_{1} \\
2 \mathrm{w}_{2} \mathrm{c} z+2 \mathrm{w}_{1} \mathrm{~s} z & 2 \mathrm{w}_{2} \mathrm{~s} z-2 \mathrm{w}_{1} \mathrm{c} z & 1-\mathrm{w}_{1}^{2}-\mathrm{w}_{2}^{2}
\end{array}\right]
$$

where $\mathrm{s}(\cdot)=\sin (\cdot)$ and $\mathrm{c}(\cdot)=\cos (\cdot)$. The matrix can be written more compactly as

$$
\mathbf{R}(\mathrm{w}, z)=\frac{1}{1+|\mathrm{w}|^{2}}\left[\begin{array}{ccc}
\operatorname{Re}\left(1+\mathrm{w}^{2}\right) e^{i z} & \operatorname{Im}\left(1+\mathrm{w}^{2}\right) e^{i z} & -2 \operatorname{Im}(\mathrm{w})  \tag{2.35}\\
\operatorname{Im}\left(1-\overline{\mathrm{w}}^{2}\right) e^{-i z} & \operatorname{Re}\left(1-\overline{\mathrm{w}}^{2}\right) e^{-i z} & 2 \operatorname{Re}(\mathrm{w}) \\
2 \operatorname{Im}\left(\mathrm{w} e^{i z}\right) & -2 \operatorname{Im}\left(\mathrm{w} e^{i z}\right. & 1-|\mathrm{w}|^{2}
\end{array}\right]
$$

### 2.4.1 Kinematic differential equations

The differential equation for (2.18) is simply $\dot{\mathbf{R}}(\mathrm{w}, z)=\mathbf{S}(\boldsymbol{\omega}) \mathbf{R}(\mathrm{w}, z)$, hence the third column of $\dot{\mathbf{R}}(\mathrm{w}, z)$ must satisfy

$$
\left[\begin{array}{c}
\dot{a}  \tag{2.36}\\
\dot{b} \\
\dot{c}
\end{array}\right]=\left[\begin{array}{ccc}
0 & \omega_{3} & -\omega_{2} \\
-\omega_{3} & 0 & \omega_{1} \\
\omega_{2} & -\omega_{1} & 0
\end{array}\right]\left[\begin{array}{l}
a \\
b \\
c
\end{array}\right]
$$

Recall that w is defined as

$$
\begin{equation*}
\mathrm{w}=\frac{b-i a}{1+c} . \tag{2.37}
\end{equation*}
$$

Differentiation of (2.37) gives

$$
\begin{equation*}
\dot{\mathrm{w}}=\frac{\dot{b}-i \dot{a}-\mathrm{w} \dot{c}}{1+c} . \tag{2.38}
\end{equation*}
$$

Using the relations in (2.31) and (2.36) gives the differential equation for w

$$
\begin{equation*}
\dot{\mathrm{w}}=-i \omega_{3} \mathrm{w}+\frac{\omega}{2}+\frac{\bar{\omega}}{2} \mathrm{w}^{2}, \tag{2.39}
\end{equation*}
$$

where

$$
\begin{equation*}
\omega=\omega_{1}+i \omega_{2}, \quad \bar{\omega}=\omega_{1}-i \omega_{2} . \tag{2.40}
\end{equation*}
$$

An alternative formulation is:

$$
\begin{align*}
& \dot{\mathrm{w}}_{1}=\omega_{3} \mathrm{w}_{2}+\omega_{2} \mathrm{w}_{1} \mathrm{w}_{2}+\frac{\omega_{1}}{2}\left(1+\mathrm{w}_{1}^{2}-\mathrm{w}_{2}^{2}\right)  \tag{2.41a}\\
& \dot{\mathrm{w}}_{2}=-\omega_{3} \mathrm{w}_{1}+\omega_{1} \mathrm{w}_{1} \mathrm{w}_{2}+\frac{\omega_{2}}{2}\left(1+\mathrm{w}_{2}^{2}-\mathrm{w}_{1}^{2}\right) \tag{2.41b}
\end{align*}
$$

To find the differential equation for $z$ we start with the scalar form of the differential equation for a rotation matrix,

$$
\begin{equation*}
\operatorname{tr}[\dot{\mathbf{R}}(\mathrm{w}, z)]=\operatorname{tr}[\mathbf{S}(\boldsymbol{\omega}) \mathbf{R}(\mathrm{w}, z)], \tag{2.42}
\end{equation*}
$$

where $\operatorname{tr}(\cdot)$ is sum of the diagonal elements of the matrix. Taking the trace of $\dot{\mathbf{R}}(\mathrm{w}, z)$ gives:

$$
\begin{align*}
\operatorname{tr}[\dot{\mathbf{R}}(\mathrm{w}, z)] & =\frac{d}{d t}(\operatorname{tr}[\mathbf{R}(\mathrm{w}, z)])=\frac{d}{d t}\left(\frac{2 \cos z+2}{1+\mathrm{w}_{1}^{2}+\mathrm{w}_{2}^{2}}-1\right) \\
& =-\frac{2 \dot{z} \sin z}{1+\mathrm{w}_{1}^{2}+\mathrm{w}_{2}^{2}}-\frac{4(1+\cos z)\left(\mathrm{w}_{1} \dot{\mathrm{w}}_{1}+\mathrm{w}_{2} \dot{\mathrm{w}}_{2}\right)}{\left(1+\mathrm{w}_{1}^{2}+\mathrm{w}_{2}^{2}\right)^{2}} \tag{2.43}
\end{align*}
$$

Combining (2.41a) and (2.41b) gives the relation

$$
\begin{equation*}
2 \frac{\mathrm{w}_{1} \dot{\mathrm{w}}_{1}+\mathrm{w}_{2} \dot{\mathrm{w}}_{2}}{\left(1+\mathrm{w}_{1}^{2}+\mathrm{w}_{2}^{2}\right)^{2}}=\omega_{1}+\mathrm{w}_{1}+\omega_{2} \mathrm{w}_{2}, \tag{2.44}
\end{equation*}
$$

which substituted into (2.43) gives the expression

$$
\begin{equation*}
\operatorname{tr}[\dot{\mathbf{R}}(\mathrm{w}, z)]=\frac{2}{1+\mathrm{w}_{1}^{2}+\mathrm{w}_{2}^{2}}\left(\dot{z} \sin z+(1+\cos z)\left(\omega_{1} \mathrm{w}_{1}+\omega_{2} \mathrm{w}_{2}\right)\right) . \tag{2.45}
\end{equation*}
$$

Expanding the right hand side of (2.42) we obtain
$\operatorname{tr}[\mathbf{S}(\boldsymbol{\omega}) \mathbf{R}(\mathrm{w}, z)]=\frac{-2}{1+\mathrm{w}_{1}^{2}+\mathrm{w}_{2}^{2}}\left((1+\cos z)\left(\omega_{1} \mathrm{w}_{1}+\omega_{2} \mathrm{w}_{2}\right)+\left(\omega_{3}-\omega_{1} \mathrm{w}_{2}+\omega_{2} \mathrm{w}_{1}\right) \sin z\right)$.
Now we can obtain the differential equation for $z$ by equating (2.46) with (2.45):

$$
\begin{equation*}
\dot{z}=\omega_{3}-\omega_{1} \mathrm{w}_{2}+\omega_{2} \mathrm{w}_{1} \tag{2.4}
\end{equation*}
$$

An equivalent expression is

$$
\begin{equation*}
\dot{z}=\omega_{3}+\frac{i}{2}(\bar{\omega} \mathrm{w}-\omega \overline{\mathrm{w}}) . \tag{2.48}
\end{equation*}
$$

To summarize the discussion above, the differential kinematic equations for the ( $\mathrm{w}, z$ ) parametrization are

$$
\begin{align*}
\dot{\mathrm{w}}_{1} & =\omega_{3} \mathrm{w}_{2}+\omega_{2} \mathrm{w}_{1} \mathrm{w}_{2}+\frac{\omega_{1}}{2}\left(1+\mathrm{w}_{1}^{2}-\mathrm{w}_{2}^{2}\right)  \tag{2.49a}\\
\dot{\mathrm{w}}_{2} & =-\omega_{3} \mathrm{w}_{1}+\omega_{1} \mathrm{w}_{1} \mathrm{w}_{2}+\frac{\omega_{2}}{2}\left(1+\mathrm{w}_{2}^{2}-\mathrm{w}_{1}^{2}\right)  \tag{2.49b}\\
\dot{z} & =\omega_{3}-\omega_{1} \mathrm{w}_{2}+\omega_{2} \mathrm{w}_{1} \tag{2.49c}
\end{align*}
$$

Alternatively they can be written more compactly as:

$$
\begin{align*}
\dot{\mathrm{w}} & =-i \omega_{3} \mathrm{w}+\frac{\omega}{2}+\frac{\bar{\omega}}{2} \mathrm{w}^{2}  \tag{2.50a}\\
\dot{z} & =\omega_{3}+\frac{i}{2}(\bar{\omega} \mathrm{w}-\omega \overline{\mathrm{w}}) \tag{2.50b}
\end{align*}
$$

Remark 2.4.1. It is straight forward to verify that the kinematic differential equations for the ( $\mathrm{w}, z$ )-parameterization can be written as

$$
\begin{align*}
\frac{d}{d t}|\mathrm{w}|^{2} & =\left(1+|\mathrm{w}|^{2}\right) \operatorname{Re}(\omega \overline{\mathrm{w}})  \tag{2.51a}\\
\dot{z} & =\omega_{3}+\operatorname{Im}(\omega \overline{\mathrm{w}}) \tag{2.51b}
\end{align*}
$$

where

$$
\begin{equation*}
|\mathrm{w}|^{2}=\mathrm{w} \overline{\mathrm{w}}=\mathrm{w}_{1}^{2}+\mathrm{w}_{2}^{2} \tag{2.52}
\end{equation*}
$$

and

$$
\begin{equation*}
\operatorname{Re}(\omega \overline{\mathrm{w}})=\omega_{1} \mathrm{w}_{1}+\omega_{2} \mathrm{w}_{2}, \quad \operatorname{Im}(\omega \overline{\mathrm{w}})=\omega_{2} \mathrm{w}_{1}-\omega_{1} \mathrm{w}_{2} \tag{2.53}
\end{equation*}
$$

Note that in (2.51a) only the real part of the term $\omega \overline{\mathrm{w}}$ appears, while in (2.51b) only the imaginary part appears. This antisymmetric property can be very useful for designing control laws.

### 2.4.2 Properties

The ( $\mathrm{w}, z$ ) parametrization has some unique properties that makes it useful in attitude control problems.

- The kinematic equations are compact and have a clear physical interpretation
- The $z$ parameter does not appear in (4.4a) and (4.4b). This means that in some applications the control problem can be decomposed into one of controlling only w and one of controlling $z$.
- A singularity appears in the parametrization when the body is upside down and consequently $\mathrm{w} \rightarrow \infty$. The equilibrium $\left(\mathrm{w}_{1}, \mathrm{w}_{2}, z\right)=(0,0,0)$ is as far as possible from the singularity.


### 2.5 Discussion

The previous sections have shown that there are many attitude parameterizations to choose from. However, the ( $\mathrm{w}, z$ )-parameterization will be used in the rest of this thesis to describe the attitude dynamics of an underactuated spacecraft. First of all this parameterization is minimal and it is easy to avoid the singularity as long as we ensure that $\left|\mathrm{w}_{1}\right|$ and $\left|\mathrm{w}_{2}\right|$ does not approach $\infty$. Secondly, if the actuator failure is about the $z$-axis, the dynamics of the unactuated axis can be decoupled from the rest of the system.

## Chapter 3

## Spacecraft dynamics

Spacecraft dynamics and the space environment is a very large and interesting subject. A brief introduction will be given in this chapter. For the interested reader, Sellers (2000) is highly recommended as an introduction to astronautics. For more in-depth information see Huges (1986) and Wertz (1999)

### 3.1 Newton-Euler equations for rigid bodies

The angular motion of a spacecraft can be modelled as an ideal rigid body. However, most spacecrafts have flexible parts like for instance antennas and solar panels. Internal effects like fuel sloshing and thermal deformations are not accounted for using a rigid body model. Nevertheless, for many problems the rigid body model is a good approximation, especially for small spacecrafts.

The well known equations for a rigid body can be written as

$$
\begin{equation*}
\vec{\tau}=\vec{M} \dot{\vec{\omega}}_{i b}+\vec{\omega}_{i b} \times\left(\vec{M} \vec{\omega}_{i b}\right) \tag{3.1}
\end{equation*}
$$

where $\vec{M}$ is the inertia dyadic of the rigid body, $\vec{\omega}_{i b}$ is the angular velocity between an inertial reference frame and the body, $\vec{\tau}$ is the torque acting on the rigid body. A detailed derivation of (3.1) can be found in Appendix B. It is more convenient to write the equations of motion in the body frame as

$$
\begin{equation*}
\mathbf{M} \dot{\boldsymbol{\omega}}_{i b}^{b}+\mathbf{S}\left(\boldsymbol{\omega}_{i b}^{b}\right) \mathbf{M} \boldsymbol{\omega}_{i b}^{b}=\boldsymbol{\tau}^{b} \tag{3.2}
\end{equation*}
$$

where

$$
\boldsymbol{\omega}_{i b}^{b}=\left[\begin{array}{lll}
\omega_{1} & \omega_{2} & \omega_{3}
\end{array}\right]^{\mathrm{T}}, \quad \boldsymbol{\tau}^{b}=\left[\begin{array}{lll}
\tau_{x} & \tau_{y} & \tau_{z}
\end{array}\right]^{\mathrm{T}}
$$

Assuming a diagonal inertia matrix $\mathbf{M}=\operatorname{diag}\left(m_{11}, m_{22}, m_{33}\right)$ the equations of motion in component form are:

$$
\begin{align*}
\dot{\omega}_{1} & =\frac{m_{22}-m_{33}}{m_{11}} \omega_{2} \omega_{3}+\frac{1}{m_{11}} \tau_{x}  \tag{3.3}\\
\dot{\omega}_{2} & =\frac{m_{33}-m_{11}}{m_{22}} \omega_{1} \omega_{3}+\frac{1}{m_{22}} \tau_{y}  \tag{3.4}\\
\dot{\omega}_{3} & =\frac{m_{11}-m_{22}}{m_{33}} \omega_{1} \omega_{2}+\frac{1}{m_{33}} \tau_{z} \tag{3.5}
\end{align*}
$$

The translational motion of a spacecraft is not considered in this thesis. However, some details can be found in Appendix B

### 3.2 Actuators

There are several types of actuators that can be used to control the orientation of a spacecraft. The actuators can be divided into three categories: thrusters, momentum exchange devices and magnetic actuators. It is common to use more than one actuator type on a spacecraft.

### 3.2.1 Thrusters

Thrusters or gas jets produce torque by expelling mass. They can be used both for attitude and position control. When used for attitude control a pair of thrusters on opposite sides of the spacecraft is needed. The main advantage of using thrusters is that they can produce an accurate and well-defined torque on demand. The main disadvantage is that a spacecraft can only carry a limited amount of propellant.

### 3.2.2 Reaction wheels

When a gyro or rotor is accelerated an angular torque is generated in the opposite direction.

$$
\begin{equation*}
\mathbf{M}_{r} \dot{\boldsymbol{\omega}}_{r}=-\mathbf{M} \dot{\boldsymbol{\omega}}_{i b}^{b} \tag{3.6}
\end{equation*}
$$

This effect is exploited when using reaction wheels to control the attitude of a spacecraft. As seen from (3.6) the wheels have to be accelerated in order to create a torque. Neglecting friction effects, the torque generated by a set of reaction wheels can be written as (Kaplan, 1976)

$$
\begin{equation*}
\boldsymbol{\tau}_{r}^{b}=\left(\frac{d}{d t} \mathbf{h}_{r}\right)^{b}+\boldsymbol{\omega}_{i b}^{b} \times \mathbf{h}_{r} \tag{3.7}
\end{equation*}
$$

where $\mathbf{h}_{3}=\left[\begin{array}{lll}h_{r x} & h_{r y} & h_{r z}\end{array}\right]^{\mathrm{T}}=\mathbf{M}_{r} \boldsymbol{\omega}_{r}$ is the wheels' total angular momentum. In component form the torque can written as

$$
\begin{align*}
\tau_{r x} & =\dot{h}_{r x}+h_{r z} \omega_{2}-h_{r y} \omega_{3}  \tag{3.8a}\\
\tau_{r y} & =\dot{h}_{r y}+h_{r x} \omega_{3}-h_{r z} \omega_{1}  \tag{3.8b}\\
\tau_{r z} & =\dot{h}_{r z}+h_{r y} \omega_{1}-h_{r x} \omega_{2} \tag{3.8c}
\end{align*}
$$

### 3.2.3 Magnetic actuators

A magnetic torquer takes advantage of the natural torque caused by Earth's magnetic field interacting with a magnet. They offer a cheap, reliable and robust way to control a spacecraft's attitude. Unfortunately they are only effective for low Earth orbit (LEO) spacecrafts and requires a complex model of the Earth's geomagnetic field. Magnetic actuators together with a gravity gradient can be used to achieve full three-axis attitude stabilization. A good example of this is the Danish satellite Ørsted (Wiśniewski and Blanke, 1999).

The torque generated by the magnetorquers can be modelled as

$$
\begin{equation*}
\boldsymbol{\tau}_{m}^{b}=\mathbf{m}^{b} \times \mathbf{B}^{b} \tag{3.9}
\end{equation*}
$$

where $\mathbf{m}^{b}$ is the magnetic dipole moment generated by the coils, and $\mathbf{B}^{b}=\left[B_{x}^{b} B_{y}^{b} B_{z}^{b}\right]^{\mathrm{T}}$ is the local geomagnetic field vector. The magnetic dipole moment is

$$
\mathbf{m}^{b}=\mathbf{m}_{x}^{b}+\mathbf{m}_{y}^{b}+\mathbf{m}_{z}^{b}=\left[\begin{array}{c}
N_{x} i_{x} A_{x}  \tag{3.10}\\
N_{y} i_{y} A_{y} \\
N_{z} i_{z} A_{z}
\end{array}\right]=\left[\begin{array}{c}
m_{x} \\
m_{y} \\
m_{z}
\end{array}\right],
$$

where $N_{k}$ is the number of windings in the magnetic coil on the axis in the $k$ direction, $i_{k}$ is the coil current and $A_{k}$ is the coil cross-section area.

### 3.3 Disturbance torques

A spacecraft is subject to small but persistent disturbance torques and forces. The main disturbance torques for a satellite orbiting Earth are briefly discussed in this section. For more details see for instance Huges (1986).

For low orbit satellites the air density is high enough to influence the satellite's attitude dynamics. The drag force also decreases the satellite's velocity, resulting in a lower altitude. Unless the the orbit is maintained using thrusters, a satellite will ultimately reenter the atmosphere. Solar radiation and particles is also a source of disturbances. Radiation can damage the on board electronics and temperature changes distort the structure of the satellite.

Several internal effects can generate disturbance torques. The electronics in the satellite may create an unwanted residual magnetic dipole. This field will interact with the Earth's geomagnetic field and generate a disturbance torque. When thrusters are used, fuel sloshing is a challenging problem. Another problem is flexible structures like antennas and solar panels.

### 3.3.1 Gravity gradient torque

The gravity gradient torque will affect a non symmetric body in the Earth's gravity field. This effect can be exploited with a gravity boom for passive stabilization. Assuming a homogeneous mass distribution of the Earth, the gravity gradient can be written as (Wertz, 1999)

$$
\begin{equation*}
\vec{\tau}_{g}=\frac{3 \mu}{R_{0}^{3}} \vec{u}_{e} \times\left(\vec{M} \vec{u}_{e}\right) \tag{3.11}
\end{equation*}
$$

where:

$$
\begin{aligned}
& \mu \text { - Earth's gravitational coefficient } \mu=3.986 \cdot 10^{14} \mathrm{~m}^{3} / \mathrm{s}^{2} \\
& R_{0} \text { - Distance from Earth's center }(\mathrm{m}) \\
& \vec{M} \text { - Spacecraft inertia matrix } \\
& \vec{u}_{e} \text { - Unit vector towards Earth's center }
\end{aligned}
$$

Writing (3.11) in the body frame yields

$$
\begin{equation*}
\boldsymbol{\tau}_{g}^{b}=\frac{3 \mu}{R_{0}^{3}} \mathbf{c}_{3}^{b} \times\left(\mathbf{M} \mathbf{c}_{3}^{b}\right) \tag{3.12}
\end{equation*}
$$

where $\mathbf{c}_{3}^{b}=\left[\begin{array}{lll}c_{23} & c_{23} & c_{33}\end{array}\right]^{\mathrm{T}}$ is the third column in the rotation matrix describing the orientation between a local reference frame and the body frame.

Assuming a diagonal inertia matrix $\mathbf{M}=\operatorname{diag}\left(m_{11}, m_{22}, m_{33}\right)$, the gravitational torque simplifies to

$$
\boldsymbol{\tau}_{g}^{b}=3 \omega_{o}^{2}\left[\begin{array}{l}
\left(m_{33}-m_{22}\right) c_{23} c_{33}  \tag{3.13}\\
\left(m_{11}-m_{33}\right) c_{33} c_{13} \\
\left(m_{22}-m_{11}\right) c_{13} c_{23}
\end{array}\right]
$$

where $\omega_{o}^{2}=\frac{\mu}{R_{0}^{3}}$.

## Chapter 4

## Control properties of an underactuated spacecraft

The underactuated rigid spacecraft has many interesting properties that makes the attitude stabilization a challenging control problem. The purpose of this chapter is to choose an adequate model and describe some of the most important properties.

### 4.1 Model

In Chapters 3 and 2 the attitude dynamics of a spacecraft were presented. To simplify the analysis it is important to choose an adequate model that is not too complicated. For the kinematics the ( $\mathrm{w}, z$ )-parameterization is chosen because of its useful properties. It is assumed that the disturbances acting on the spacecraft are ignorable and no gravity gradient is present. It is also assumed that the torque can be controlled directly, for instance using thrusters.

Consider the case when there is no available control torque about the third principal axis, due to for instance an actuator failure. We then have an underactuated spacecraft. In order to simplify the rigid body dynamics in (3.3) the following feedback transformations are introduced:

$$
\begin{align*}
& \tau_{a}=\frac{m_{22}-m_{33}}{m_{11}} \omega_{2} \omega_{3}+\frac{1}{m_{11}} \tau_{x}  \tag{4.1}\\
& \tau_{b}=\frac{m_{33}-m_{11}}{m_{22}} \omega_{1} \omega_{3}+\frac{1}{m_{22}} \tau_{y} \tag{4.2}
\end{align*}
$$

The complete attitude dynamics for an underactuated spacecraft can then be written as

$$
\begin{align*}
\dot{\mathrm{w}}_{1} & =\omega_{3} \mathrm{w}_{2}+\omega_{2} \mathrm{w}_{1} \mathrm{w}_{2}+\frac{\omega_{1}}{2}\left(1+\mathrm{w}_{1}^{2}-\mathrm{w}_{2}^{2}\right)  \tag{4.4a}\\
\dot{\mathrm{w}}_{2} & =-\omega_{3} \mathrm{w}_{1}+\omega_{1} \mathrm{w}_{1} \mathrm{w}_{2}+\frac{\omega_{2}}{2}\left(1+\mathrm{w}_{2}^{2}-\mathrm{w}_{1}^{2}\right)  \tag{4.4b}\\
\dot{z} & =\omega_{3}-\omega_{1} \mathrm{w}_{2}+\omega_{2} \mathrm{w}_{1}  \tag{4.4c}\\
\dot{\omega}_{1} & =\tau_{a}  \tag{4.4d}\\
\dot{\omega}_{2} & =\tau_{b}  \tag{4.4e}\\
\dot{\omega}_{3} & =c_{3} \omega_{1} \omega_{2}, \tag{4.4f}
\end{align*}
$$

where $c_{3}=\frac{m_{11}-m_{22}}{m_{33}}$. It is assumed that $c_{3} \neq 0$.

Definition 4.1 (Axi-symmetric rigid body). When $c_{3}=0 \Rightarrow \dot{\omega}_{3}=0$ the rigid body is axi-symmetric.

Remark 4.1.1. A real spacecraft can never be completely axi-symmetric, implying that even if $c_{3}$ is very small there will be a slow rotation about the symmetric axis.

Remark 4.1.2. The model (4.4) can be considered as a deep-space probe with only two pairs of thrusters.

The underactuated rigid spacecraft has many similarities with other underactuated systems. Several underactuated vehicles can in fact be described by the general model (Pettersen, 1996):

$$
\begin{align*}
\mathbf{M} \dot{\boldsymbol{\nu}}+\mathbf{C}(\boldsymbol{\nu}) \boldsymbol{\nu}+\mathbf{D}(\boldsymbol{\nu}) \boldsymbol{\nu}+\mathbf{g}(\boldsymbol{\eta}) & =\left[\begin{array}{l}
\boldsymbol{\tau} \\
0
\end{array}\right]  \tag{4.5a}\\
\dot{\boldsymbol{\eta}} & =\mathbf{J}(\boldsymbol{\eta}) \boldsymbol{\nu} \tag{4.5b}
\end{align*}
$$

where

$$
\boldsymbol{\eta} \in \mathbb{R}^{n_{1}}, \boldsymbol{\nu} \in \mathbb{R}^{n_{2}}, n_{1} \leq n_{2}, \boldsymbol{\tau} \in \mathbb{R}^{m}, m<n_{2}
$$

The vector $\boldsymbol{\nu}$ denotes linear and angular velocities, and $\boldsymbol{\eta}$ denotes the position and orientation of the vehicle. Gravitational and buoyant forces and torques are denoted by $\mathbf{g}(\boldsymbol{\eta}), \mathbf{M}$ is the inertia matrix, $\mathbf{C}(\boldsymbol{\nu})$ is the Coriolis and centripetal matrix and $\mathbf{D}(\boldsymbol{\nu})$ is the damping matrix. Some examples of vehicles described by (4.5) are underactuated surface vessels, underwater vehicles and spacecrafts.

The spacecraft model (4.4)can be written in the form (4.5) by setting

$$
\begin{gathered}
\boldsymbol{\nu}=\boldsymbol{\omega}=\left[\omega_{1}, \omega_{2}, \omega_{3}\right]^{\mathrm{T}}, \boldsymbol{\eta}=\left[\mathrm{w}_{1}, \mathrm{w}_{2}, z\right]^{\mathrm{T}}, \boldsymbol{\tau}=\left[\tau_{a}, \tau_{b}\right]^{T} \\
\mathbf{D}(\boldsymbol{\nu})=\mathbf{0}, \\
\mathbf{C}(\boldsymbol{\nu})=\mathbf{S}(\boldsymbol{\omega}) \mathbf{M}, \mathbf{J}(\boldsymbol{\eta})=\left[\begin{array}{ccc}
\frac{1}{2}\left(1+\mathrm{w}_{1}^{2}-\mathrm{w}_{2}^{2}\right) & \mathrm{w}_{1} \mathrm{w}_{2} & \mathrm{w}_{2} \\
\mathrm{w}_{1} \mathrm{w}_{2} & \frac{1}{2}\left(1+\mathrm{w}_{2}^{2}-\mathrm{w}_{1}^{2}\right) & -\mathrm{w}_{1} \\
-\mathrm{w}_{2} & \mathrm{w}_{1} & 1
\end{array}\right]
\end{gathered}
$$

Remark 4.1.3. The fundamental difference between the underactuated spacecraft model and the general model (4.5), is the lack of a damping term, i.e, $\mathbf{D}(\boldsymbol{\nu})=\mathbf{0}$. This turns out to be a major disadvantage when designing stabilizing control laws.

Remark 4.1.4. A spacecraft with a gravity gradient will experience gravitational torques, hence $\mathbf{q}(\eta) \neq \mathbf{0}$.

### 4.2 Stabilizability

Property 4.1. There exists no continuous time-invariant state feedback that renders the system (4.4) asymptotically stable about the origin.

Proof. (Based on Proposition 2.3 in Pettersen (1996)) Consider the mapping $\mathbf{f}(\boldsymbol{\eta}, \boldsymbol{\omega}, \boldsymbol{\tau}): \mathbb{R}^{n_{1}} \times$ $\mathbb{R}^{n_{2}} \times \mathbb{R}^{m} \rightarrow \mathbb{R}^{n_{1}+n_{2}}$ defined by

$$
\mathbf{f}(\boldsymbol{\eta}, \boldsymbol{\omega}, \boldsymbol{\tau})=\left[\begin{array}{c}
\mathbf{J}(\boldsymbol{\eta}) \boldsymbol{\omega}  \tag{4.6}\\
-\mathbf{M}^{-1} \mathbf{S}(\boldsymbol{\omega}) \mathbf{M} \boldsymbol{\omega}+\mathbf{M}^{-1}\left[\begin{array}{l}
\boldsymbol{\tau} \\
0
\end{array}\right]
\end{array}\right]
$$

To prove Property 4.1 we must show that $\mathbf{f}(\cdot)$ is not locally surjective. Consider a point $\varepsilon$ in $\mathbb{R}^{n_{1}+n_{2}}$ of the form

$$
\boldsymbol{\varepsilon}=\left[\begin{array}{c}
\mathbf{0}_{n_{1} \times 1}  \tag{4.7}\\
\mathbf{M}^{-1}\left[\begin{array}{c}
\alpha_{1} \\
\alpha_{2} \\
\beta
\end{array}\right]
\end{array}\right]
$$

where $\left(\alpha_{1}, \alpha_{2}, \beta\right) \in \mathbb{R}$ and non-zero. Points on the form $\varepsilon$ exist in any neighborhood of $\mathbf{0}$ in $\mathbb{R}^{n_{1}+n_{2}}$. For $\mathbf{f}(\cdot)$ to be surjective then for any $\boldsymbol{\varepsilon} \in \mathbb{R}^{n_{1}+n_{2}}$ there exists an $\boldsymbol{\delta} \in \mathbb{R}^{n_{1}+n_{2}} \times \mathbb{R}^{m}$ for which $\mathbf{f}(\boldsymbol{\delta})=\boldsymbol{\varepsilon}$. However, the solution of $\mathbf{f}(\boldsymbol{\eta}, \boldsymbol{\omega}, \boldsymbol{\tau})=\boldsymbol{\varepsilon}$ implies

$$
\left[\begin{array}{c}
\tau_{a}  \tag{4.8}\\
\tau_{b} \\
0
\end{array}\right]=\left[\begin{array}{c}
\alpha_{1} \\
\alpha_{2} \\
\beta
\end{array}\right]
$$

since $\mathbf{J}(\boldsymbol{\eta})$ has full rank and therefore $\boldsymbol{\omega}=\mathbf{0}$. Clearly (4.8) has no solution. From Theorem A. 2 it then follows that Property 4.1 is true.

Remark 4.2.1. In Pettersen (1996) a more general result was derived. It was in fact shown that there exists no continuous nor discontinous pure-state feedback law that makes the origin of (4.5) asymptotically stable if $\mathbf{g}^{u}(\boldsymbol{\eta})$ has a zero element. The vector $\mathbf{g}^{u}(\boldsymbol{\eta})$ is the elements of $\mathbf{g}(\boldsymbol{\eta})$ corresponding to the underactuated dynamics .

The following property is a result of the fact that a smooth nonlinear control system is exponentially stabilizable using smooth feedback only if its linearization about the origin is stabilizable

Property 4.2. The system (4.4) can not be exponentially stabilized by using smooth feedback laws. The asymptotic rate of convergence to zero is only polynomial in the worst case when the control laws are smooth.

Proof. See for instance Reyhanoglu (1996) and Morin and Samson (1997)

Property 4.2 explains why there has been an emphasis on discontinous and almost continuous control laws for the stabilization of underactuated systems in the control community the latest years.

### 4.3 Investigation of the underactuated dynamics

Some important insight can be gained about the system (4.4) by considering the case when $\left(\mathrm{w}_{1}, \mathrm{w}_{2}\right) \approx(0,0)$.

$$
\begin{align*}
\dot{\mathrm{w}}_{1} & =\frac{1}{2} \omega_{1}  \tag{4.9a}\\
\dot{\mathrm{w}}_{2} & =\frac{1}{2} \omega_{2}  \tag{4.9b}\\
\dot{z} & =\omega_{3}  \tag{4.9c}\\
\dot{\omega}_{3} & =c_{3} \omega_{1} \omega_{2}  \tag{4.9d}\\
\dot{\omega}_{1} & =\tau_{a}  \tag{4.9e}\\
\dot{\omega}_{2} & =\tau_{b} \tag{4.9f}
\end{align*}
$$

First of all we see from (4.9c) and (4.9d) that we have no direct control of $z$ and $\omega_{3}$. However, $z$ can be manipulated indirectly through the term $c_{3} \omega_{1} \omega_{2}$ in (4.9d). If the spacecraft is axi-symmetric, $c_{3}=0$, we have no control at all.

Assume that it is possible to manipulate the angular velocities directly. Consider the time varying, periodic controllers

$$
\begin{equation*}
\omega_{1}=-\omega_{3} \sin t, \quad \omega_{2}=\cos t \tag{4.10}
\end{equation*}
$$

Insertion of the controllers into (4.9d) gives

$$
\begin{equation*}
\dot{\omega}_{3}=-c_{3} \omega_{3} \sin ^{2} t \tag{4.11}
\end{equation*}
$$

which has the average value

$$
\begin{equation*}
\dot{\bar{\omega}}_{3}=-\frac{1}{T} \int_{0}^{T} \omega_{3} \sin ^{2}(t) d t=-\frac{1}{2} \omega_{3} \tag{4.12}
\end{equation*}
$$

with $T=\pi$. This means that it in average is possible to control $\omega_{3}$ by using time-varying periodic controllers. This is the basic and intuitive idea behind many controllers for underactuated and nonholonomic systems ${ }^{1}$. Several other control strategies exist, but in order to circumvent Brockett's necessary condition they must be time-varying or discontinous. See for instance Kolmanovsky and McClamroch (1995) for a comparison of different control strategies for nonholonomic systems.

[^1]
## Chapter 5

## Extending the results of Mazenc et al. (2002) to the spacecraft attitude stabilization problem

In Mazenc et al. (2002) the the open problem of determining explicit expressions of smooth time-varying periodic state feedbacks, which render the origin of an underactuated surface vessel globally uniformly asymptotically stable (GUAS), was solved. The purpose of this chapter is to describe the method used in Mazenc et al. (2002) and investigate if it can be extended to the stabilization problem of an underactuated rigid spacecraft.

### 5.1 Global uniform asymptotic stabilization of an underactuated surface vessel

In the subsequent sections the model of an underactuated surface vessel is presented and it is it shown step by step how the stabilization problem was solved in Mazenc et al. (2002).

### 5.1.1 Model of an underactuated surface vessel

The dynamics of an underactuated surface vessel can be described by the nonlinear model:

$$
\begin{align*}
\dot{u} & =\frac{m_{22}}{m_{11}} v r-\frac{d_{11}}{m_{11}} u+\frac{1}{m_{11}} \tau_{1}  \tag{5.1a}\\
\dot{v} & =-\frac{m_{11}}{m_{22}} u r-\frac{d_{22}}{m_{22}} v  \tag{5.1b}\\
\dot{r} & =\frac{m_{11}-m_{22}}{m_{33}} u v-\frac{d_{33}}{m_{33}} r+\frac{1}{m_{33}} \tau_{3} \tag{5.1c}
\end{align*}
$$

where $u, v$ and $r$ are the velocities in surge, sway and yaw respectively, $\tau_{1}$ is the surge control force and $\tau_{3}$ the yaw control moment. The parameters $m_{i i}$ and $d_{i i}$ are given by the ships inertia, added mass effects and hydrodynamic damping. Note that there is no available control in sway. The kinematics of the ship are

$$
\begin{align*}
\dot{x} & =\cos (\psi) u-\sin (\psi) v  \tag{5.2a}\\
\dot{y} & =\sin (\psi) u+\cos (\psi) v  \tag{5.2b}\\
\dot{\psi} & =r \tag{5.2c}
\end{align*}
$$

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where $x$ and $y$ is the ship's position, and $\psi$ is the orientation.

### 5.1.2 Stabilization

Step I. Model simplification To obtain simpler, polynomial equations a global coordinate transformation was introduced and the dynamics simplified ${ }^{1}$. The resulting system was:

$$
\begin{align*}
\dot{z}_{1} & =u+z_{2} r  \tag{5.3a}\\
\dot{z}_{2} & =v-z_{1} r  \tag{5.3b}\\
\dot{z}_{3} & =r  \tag{5.3c}\\
\dot{u} & =\tau_{u}  \tag{5.3d}\\
\dot{v} & =-c u r-d v  \tag{5.3e}\\
\dot{r} & =\tau_{r} \tag{5.3f}
\end{align*}
$$

Step II The underactuated variable $v$ was removed from the z -subsystem by introducing the variable

$$
\begin{equation*}
Z_{2}=z_{2}+\frac{v}{d} \tag{5.4}
\end{equation*}
$$

and a stabilizing term was introduced in the $z_{1}$ equation by a change of coordinate

$$
\begin{equation*}
u=-\frac{d}{c} z_{1}-\frac{d}{c} \mu \tag{5.5}
\end{equation*}
$$

The resulting model was

$$
\begin{align*}
\dot{z}_{1} & =-\frac{d}{c} z_{1}-\frac{d}{c} \mu+Z_{2} r-\frac{v}{d} r  \tag{5.6a}\\
\dot{Z}_{2} & =\mu r  \tag{5.6b}\\
\dot{z}_{3} & =r  \tag{5.6c}\\
\dot{v} & =-d v+d\left(z_{1}+\mu\right) r  \tag{5.6d}\\
\dot{\mu} & =\tau_{\mu}  \tag{5.6e}\\
\dot{r} & =\tau_{r} \tag{5.6f}
\end{align*}
$$

Note that the system (5.6) has the cascade structure shown in Figure 5.1, with:

$$
\begin{align*}
\xi & =\left[Z_{2}, z_{3}, \mu, r\right]^{\mathrm{T}}  \tag{5.7}\\
z & =\left[z_{1}, v\right]^{\mathrm{T}}  \tag{5.8}\\
u & =\left[\tau_{\mu}, \tau_{r}\right]^{\mathrm{T}} \tag{5.9}
\end{align*}
$$

The purpose of the various transformations was to make the system amenable for partial-state feedback designs. In partial-state feedback designs only the $\xi$-subsystem state is used for feedback and the interconnection between the $\xi$ - and $z$-subsystem is considered as a disturbance. The cascade system can be written as

$$
\begin{align*}
\dot{z} & =f(z)+\psi(z, \xi)  \tag{5.10}\\
\dot{\xi} & =a(\xi, u) \tag{5.11}
\end{align*}
$$

[^2]

Figure 5.1: A cascade system
where $\psi(z, \xi)$ is the interconnection term. In some cases the stabilization of the $\xi$-subsystem ensures the stabilization of the entire cascade, however this imposes severe growth restrictions on the interconnection term $\psi(z, \xi)$. If the growth of $\psi(z, \xi)$ in $z$ is faster than linear it is a structural obstacle to both global and semi global stabilization. For more details about cascaded systems, see for instance Sepulchre et al. (1997).

Step III As a result of the cascade structure it was shown that the total system (5.6) is globally uniformly asymptotically stabilized by any control law which globally uniformly asymptotically stabilizes the subsystem:

$$
\begin{align*}
\dot{Z}_{2} & =\mu r  \tag{5.12a}\\
\dot{z}_{3} & =r  \tag{5.12b}\\
\dot{\mu} & =\tau_{\mu}  \tag{5.12c}\\
\dot{r} & =\tau_{r} \tag{5.12d}
\end{align*}
$$

This was proved by adapting Property 4.11 in Sepulchre et al. (1997) to the case of time-varying systems.

Step IV To find controllers for the subsystem (5.12) the backstepping technique was used by first considering $\mu$ and $r$ as virtual inputs to the system

$$
\begin{align*}
\dot{Z}_{2} & =\mu_{f} r_{f}  \tag{5.13a}\\
\dot{z}_{3} & =r_{f} \tag{5.13b}
\end{align*}
$$

According to Brockett's theorem (Brockett, 1985) there exist no continuous time invariant feedbacks which locally asymptotically stabilizes the system (5.13). However, it is well known that the system is controllable and can be asymptotically stabilized by time-varying feedback. To obtain explicit expressions of such feedbacks, a time-varying change of variable was performed

$$
\begin{equation*}
Z_{3}=z_{3}+k_{2} \cos (t) Z_{2} \tag{5.14}
\end{equation*}
$$

resulting in the system

$$
\begin{align*}
& \dot{Z}_{2}=\mu_{f} r_{f}  \tag{5.15}\\
& \dot{Z}_{3}=r_{f}\left(1+k_{2} \cos (t) \mu_{f}\right)-k_{2} \sin (t) Z_{2} \tag{5.16}
\end{align*}
$$

By choosing $r_{f}$ and $\mu_{f}$ as

$$
\begin{equation*}
\mu_{f}=-\frac{\sin (t) Z_{2}^{2}}{2\left(0.001+Z_{2}^{2}\right.}, \quad r_{f}=\frac{-k_{3} Z_{3}+k_{2} \sin (t) Z_{2}}{1+k_{2} \cos (t) \mu_{f}}, \tag{5.17}
\end{equation*}
$$

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it was shown that the Lyapunov function $V_{1}=Z_{2}^{2}+Z_{3}^{2}$ satisfies

$$
\begin{equation*}
\dot{V}_{1} \leq-\sin ^{2}(t) W_{1}\left(Z_{2}, Z_{3}\right)-\frac{k_{3}}{2} Z_{3}^{2} \tag{5.18}
\end{equation*}
$$

where $W_{1}\left(Z_{2}, Z_{3}\right)$ is of class $\mathcal{K}_{\infty}{ }^{2}$. Equation (5.18) is only negative semi definite, but by exploiting recent results in Mazenc (2003) of the construction of strict Lyapunov functions for time-varying systems, it was shown that it was possible to find a strict Lyapunov function $V_{2}$ that satisfies

$$
\begin{gather*}
V_{1} \leq V_{2} \leq\left(2+k_{3}\right) V_{1}  \tag{5.19}\\
\dot{V}_{2} \leq-\gamma\left(V_{1}\right)<0 \tag{5.20}
\end{gather*}
$$

where $\gamma\left(V_{1}\right)$ is of class $\mathcal{K}_{\infty}$. This proves that the feedbacks $r_{f}$ and $\mu_{f}$ globally uniformly asymptotically stabilizes the subsystem (5.15) (See Theorem A.1).

Step V Backstepping was applied to obtain explicit expressions of global uniform asymptotically stabilizing feedbacks for (5.12). The feedbacks were smooth time-varying periodic state feedbacks. Because of the knowledge of a strict Lyapunov function for (5.15) it is possible to exploit robustness backstepping results to determine reasonably simple expressions of stabilizing feedbacks.

### 5.2 Comparison

In the previous section it was shown how the stabilization problem of a underactuated surface vessel can be solved. In this section it will be investigated if it is possible to extend the method to the stabilization problem of an underactuated rigid spacecraft.

### 5.2.1 Model comparison

Table 5.1 shows a side by side comparison of the simplified models of an underactuated surface vessel and an underactuated rigid spacecraft. The dynamics of both systems have a similar structure (compare (5.1) and (3.3)). However, while the surface vessel has damping terms, the rigid spacecraft has no damping at all. This is not a surprise since a spacecraft operates in an environment where there is nearly no friction. Damping is advantageous in stabilization problems since it extracts energy from the system. When no damping is present, energy is conserved in the system and typically results in oscillations which must be damped using actuators.

The kinematics of the surface vessel describe the position and course, while the kinematics of the spacecraft describe the orientation only. The kinematic equations for the surface vessel are much simpler that for the spacecraft. The $(\mathrm{w}, z)$-parameterization is used to describe the kinematics of the spacecraft. If the Euler parameters are used instead, it will result in simpler equations, but with the additional cost of adding an extra parameter. Another difference is that the vessels kinematics are global and nonsingular. This is not the case for the ( $\mathrm{w}, z$ )parameterization since it is singular when $|\mathrm{w}| \rightarrow \infty$.

[^3]Table 5.1: Comparison of the model of an underactuated surface vessel (left) and the model of an underactuated rigid spacecraft (right).

$$
\begin{aligned}
\dot{z}_{1} & =u+z_{2} r & \dot{\mathrm{w}}_{1} & =\omega_{3} \mathrm{w}_{2}+\omega_{2} \mathrm{w}_{1} \mathrm{w}_{2}+\frac{\omega_{1}}{2}\left(1+\mathrm{w}_{1}^{2}-\mathrm{w}_{2}^{2}\right) \\
\dot{z}_{2} & =v-z_{1} r & \dot{\mathrm{w}}_{2} & =-\omega_{3} \mathrm{w}_{1}+\omega_{1} \mathrm{w}_{1} \mathrm{w}_{2}+\frac{\omega_{2}}{2}\left(1+\mathrm{w}_{2}^{2}-\mathrm{w}_{1}^{2}\right) \\
\dot{z}_{3} & =r & \dot{z} & =\omega_{3}-\omega_{1} \mathrm{w}_{2}+\omega_{2} \mathrm{w}_{1} \\
\dot{u} & =\tau_{u} & \dot{\omega} & =\tau_{a} \\
\dot{v} & =-c u r-d v & \dot{\omega}_{2} & =\tau_{b} \\
\dot{r} & =\tau_{r} & \dot{\omega}_{3} & =c_{3} \omega_{1} \omega_{2}
\end{aligned}
$$

To summarize the discussion above, the models of the surface vessel and rigid spacecraft have a similar structure, but the kinematic equations of the surface vessel are much simpler. Additionally, the rigid spacecraft has no damping terms that can be exploited in controller designs.

### 5.2.2 Stabilization analysis

One of the crucial steps in Mazenc et al. (2002) was to transform the system to the cascaded structure in Figure 5.1. Many hours have been used to try to transform the underactuated spacecraft model to a similar structure. However, because of the lack of damping in the underactuated variable $\omega_{3}$, all attempts have failed. The consequence is that it is not possible to isolate a subsystem that ensures the stability of the whole system. All of the states must then be considered when designing control laws, as opposed to the surface vessel where only four states are needed. A simplification of the spacecraft equations can however be achieved by ignoring the $z$ equation since it does not affect the rest of the system.

Remark 5.2.1. It is possible to introduce damping in a spacecraft using devices called dampers that changes angular momentum by absorbing energy. One simple type of momentum damper consists of a small ball in a circular tube filled with highly viscous fluid. As a spacecraft rotates, some of its momentum is contained in the the ball that moves inside the tube. Friction between the ball and the fluid in the tube converts some of the momentum into heat that slowly dissipates throughout the spacecraft (Sellers, 2000). Such devices are often used in spinning spacecrafts to remove wobbling about the spin axis. The use of dampers raise many practical questions and was therefore not considered in this thesis. However, it may be regarded as future work.

In step IV the sub-problem of stabilizing $Z_{2}$ and $z_{3}$ using $\mu$ and $r$ as virtual inputs is considered. The same approach can be used in the spacecraft stabilization problem, i.e., $\omega_{1}$ and $\omega_{2}$ are

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considered as virtual inputs to the system:

$$
\begin{align*}
\dot{\mathrm{w}}_{1} & =\omega_{3} \mathrm{w}_{2}+\omega_{2} \mathrm{w}_{1} \mathrm{w}_{2}+\frac{\omega_{1}}{2}\left(1+\mathrm{w}_{1}^{2}-\mathrm{w}_{2}^{2}\right)  \tag{5.21}\\
\dot{\mathrm{w}}_{2} & =-\omega_{3} \mathrm{w}_{1}+\omega_{1} \mathrm{w}_{1} \mathrm{w}_{2}+\frac{\omega_{2}}{2}\left(1+\mathrm{w}_{2}^{2}-\mathrm{w}_{1}^{2}\right)  \tag{5.22}\\
\dot{z} & =\omega_{3}-\omega_{1} \mathrm{w}_{2}+\omega_{2} \mathrm{w}_{1}  \tag{5.23}\\
\dot{\omega_{3}} & =c_{3} \omega_{1} \omega_{2} \tag{5.24}
\end{align*}
$$

This technique is for instance used in Morin et al. (1995) and Behal et al. (2002). In the case of an axi-symmetric spacecraft, the problem is reduced to finding a pure kinematic controller for the sub-system:

$$
\begin{align*}
\dot{\mathrm{w}}_{1} & =\omega_{2} \mathrm{w}_{1} \mathrm{w}_{2}+\frac{\omega_{1}}{2}\left(1+\mathrm{w}_{1}^{2}-\mathrm{w}_{2}^{2}\right)  \tag{5.25}\\
\dot{\mathrm{w}}_{2} & =\omega_{1} \mathrm{w}_{1} \mathrm{w}_{2}+\frac{\omega_{2}}{2}\left(1+\mathrm{w}_{2}^{2}-\mathrm{w}_{1}^{2}\right)  \tag{5.26}\\
\dot{z} & =-\omega_{1} \mathrm{w}_{2}+\omega_{2} \mathrm{w}_{1} \tag{5.27}
\end{align*}
$$

Kinematic controllers for the axi-symmetric spacecraft are for instance derived in Tsiotras and Luo (2000). Once a controller for the sub-system is found, it is relatively easy to find a controller for the complete system using for instance backstepping.

Remark 5.2.2. The dynamics of $\omega_{1}$ and $\omega_{2}$ can be written as

$$
\dot{\omega}_{1}=\tau_{a}, \quad \dot{\omega}_{2}=\tau_{b}
$$

meaning that $\dot{\omega}_{1}$ and $\dot{\omega}_{2}$ can be considered as control inputs. This corresponds to the classical situation where integrators are added at the input level.

One of the particular features of the control design in Mazenc et al. (2002) is the method used for construction of strict Lyapunov functions for the time varying system. The method is very usefull if feedbacks are found that satisfies Theorem A.3. In Section 6.2 this method is exploited to design control laws that globally uniformly asymptotically stabilizes the angular velocities of a rigid spacecraft. Unfortunately it has proven to be hard to find such feedbacks for the complete attitude equations.

### 5.2.3 Summary

The results of Mazenc et al. (2002) can not be directly applied to the spacecraft stabilization problem. It fails because the spacecraft model has no damping and it is therefore difficult to transform the system into a less difficult subproblem. However, several tools and methods are provided that can be very usefull for the attitude stabiliation problem. Especially the method for the construction of strict Lyapunov functions.

## Chapter 6

## Attitude stabilization of an underactuated rigid spacecraft

The purpose of this chapter is to attempt to solve the open problem of determining explicit expressions of continous feedbacks which render the origin of an underactuated rigid spacecraft globally asymptotically stable. The task is somewhat ambitious, but an attempt is made to gradually solve the main problem by first solving the less difficult subproblems: spin-axis stabilization, angular velocity stabilization and partial attitude stabilization. Each subploblem has a higher degree of difficulty.

### 6.1 Spin-axis stabilization

It is well known that the attitude of an underactuated rigid spacecraft can not be stabilized by a time-invariant smooth state feedback. However, stabilization to an equilibrium manifold is possible. This was shown in Byrnes and Isidori (1991) where the attitude was stabilized to a circular attractor about the origin. Similar results were derived for the underactuated surface vessel in Pettersen (1996). In this section it is shown that stabilization to an equilibrium manifold can be achieved with very simple and elegant control laws when using the ( $\mathrm{w}, z$ )parameterization.

In the spin-axis stabilization problem we try to design control laws that achieve $\mathrm{w}_{1}=\mathrm{w}_{2}=$ $\omega_{1}=\omega_{2}=0$. By ignoring the spin about the underactuated axis the spacecraft equations can be reduced to:

$$
\begin{align*}
& \dot{\mathrm{w}}_{1}=\omega_{3}(t) \mathrm{w}_{2}+\omega_{2} \mathrm{w}_{1} \mathrm{w}_{2}+\frac{\omega_{1}}{2}\left(1+\mathrm{w}_{1}^{2}-\mathrm{w}_{2}^{2}\right)  \tag{6.1a}\\
& \dot{\mathrm{w}}_{2}=-\omega_{3}(t) \mathrm{w}_{1}+\omega_{1} \mathrm{w}_{1} \mathrm{w}_{2}+\frac{\omega_{2}}{2}\left(1+\mathrm{w}_{2}^{2}-\mathrm{w}_{1}^{2}\right)  \tag{6.1b}\\
& \dot{\omega}_{1}=\tau_{a}  \tag{6.1c}\\
& \dot{\omega}_{2}=\tau_{b} \tag{6.1d}
\end{align*}
$$

The angular velocity $\omega_{3}$ is considered as a time-varying parameter, but due to the unique properties of the ( $\mathrm{w}, z$ )-parametrization, $\omega_{3}$ can be ignored in the following analysis.

Proposition 6.1. The choice of the linear feedback control laws

$$
\begin{align*}
\tau_{a} & =-\kappa_{1} \omega_{1}-\kappa_{2} \mathrm{w}_{1}  \tag{6.2}\\
\tau_{b} & =-\kappa_{1} \omega_{2}-\kappa_{2} \mathrm{w}_{2} \tag{6.3}
\end{align*}
$$

with $\kappa_{1}>0$ and $\kappa_{2}>0$, globally asymptotically stabilizes the system (6.1) to the equilibrium manifold $\mathrm{w}_{1}=\mathrm{w}_{2}=\omega_{1}=\omega_{2}=0$.

Proof. In Tsiotras and Longuski (1994) it was shown that Proposition 6.1 is true for the axisymmetric spacecraft. However, the following analysis shows that it can be extended to the non-symmetric case as well. Consider the LPV function

$$
\begin{equation*}
V=\frac{1}{2}\left(\omega_{1}^{2}+\omega_{2}^{2}\right)+\kappa_{2} \ln \left(1+\mathrm{w}_{1}^{2}+\mathrm{w}_{2}^{2}\right) \tag{6.4}
\end{equation*}
$$

Differentiation along the trajectories of (6.1) gives

$$
\begin{equation*}
\dot{V}=\omega_{1} \tau_{a}+\omega_{2} \tau_{b}+\kappa_{2}\left(\omega_{1} \mathrm{w}_{1}+\omega_{2} \mathrm{w}_{2}\right) \tag{6.5}
\end{equation*}
$$

where we have used (2.51a). Note that $\omega_{3}$ does not appear in the expression for $\dot{V}$. Insertion of the control laws (6.2) and (6.3) yields

$$
\begin{equation*}
\dot{V}=-\kappa_{1}\left(\omega_{1}^{2}+\omega_{2}^{2}\right) \leq 0 \tag{6.6}
\end{equation*}
$$

The function $\dot{V}$ is only negative semi definite, so a more careful analyzis is necessary. If $\dot{V}=0$ then $\omega_{1}=\omega_{2}=\dot{\omega}_{1}=\dot{\omega}_{2}=0$. Insertion of the controllers into (6.1c) and (6.1d) yields

$$
\begin{align*}
\dot{\omega}_{1} & =-\kappa_{1} \omega_{1}-\kappa_{2} \mathrm{w}_{1}  \tag{6.7}\\
\dot{\omega}_{2} & =-\kappa_{1} \omega_{2}-\kappa_{2} \mathrm{w}_{2} \tag{6.8}
\end{align*}
$$

Clearly $\dot{\omega}_{1}=\dot{\omega}_{2}=0$ implies $\mathrm{w}_{1}=\mathrm{w}_{2}=0$. Since $V$ is radially unbounded, it follows from LaSalle's theorem that the equilibrium manifolds is globally asymptotically stable.

Remark 6.1.1. It is important to clerify that Proposition 6.1 is true if and only if the inital attitude of the spacecraft is nonsingular. However, if such an initial singular orientation is detected, it can be avoided by switching on one of the actuators for an arbitrary short period of time before the controllers are activated.

### 6.1.1 Simulation with the spin-axis controller

The control laws (6.2) and (6.3) was simulated with initial conditions

$$
\left[\mathrm{w}_{1}(0), \mathrm{w}_{2}(0), z(0), \omega_{1}(0), \omega_{2}(0), \omega_{3}(0)\right]^{\mathrm{T}}=[-1,1,1.28,0.1,-0.01,-1.2]^{\mathrm{T}}
$$

and parameters

$$
c_{3}=0.2, \kappa_{1}=\kappa_{2}=2
$$

Figure 6.1 and 6.2 shows the time evolution of the orientation and the angular velocities. The trajectory in the $\mathrm{w}_{1} \mathrm{w}_{2}$ plane is shown in Figure 6.4 along with the time evolution of the $z$-parameter. The simulations clearly show that stabilization to the equilibrium manifold is quickly achieved and that $\omega_{3}$ converge to a constant value.


Figure 6.1: The time evolution of the orientation $\mathrm{w}_{1}(-), \mathrm{w}_{2}(--)$ when using the spin-axis controller


Figure 6.2: The time evolution of the angular velocities $\omega_{1}(-), \omega_{2}(--), \omega_{3}(-\cdot)$ when using the spin-axis controller


Figure 6.3: The time evolution of the torque controls $\tau_{a}(-), \tau_{b}(--)$ when using the spin-axis controller.



Figure 6.4: Upper plot shows the trajectory in the $\mathrm{w}_{1} \mathrm{w}_{2}$ plane. Lower plot shows the time evolution of $z$.

### 6.2 Stabilization of the angular velocities

In the previous section it was shown that it is relatively easy to stabilize the spacecraft to a uniform rotation about the underactuated axis. In this section the more difficult problem of angular velocity stabilization is solved.

$$
\begin{align*}
\dot{\omega}_{1} & =\tau_{a}  \tag{6.9a}\\
\dot{\omega}_{2} & =\tau_{b}  \tag{6.9b}\\
\dot{\omega}_{3} & =c_{3} \omega_{1} \omega_{2} \tag{6.9c}
\end{align*}
$$

Proposition 6.2. Consider the angular velocity system (6.9). Let $k_{1}, k_{2}, k_{a}$ and $k_{b}$ be strictly positive parameters. Then the system is globally uniformly asymptotically stabilized by the feedbacks

$$
\begin{align*}
\tau_{a} & =-k_{a}\left(\omega_{1}-\omega_{1 d}\right)+\dot{\omega}_{1 d}-\delta \omega_{2}  \tag{6.10a}\\
\tau_{b} & =-k_{b}\left(\omega_{2}-\omega_{2 d}\right)+\dot{\omega}_{2 d}-\delta \omega_{1 d} \tag{6.10b}
\end{align*}
$$

where

$$
\begin{align*}
\omega_{1 d} & =-\frac{\omega_{3}^{2} \sin t}{\beta+\omega_{3}^{2}}, \quad \beta \ll 1  \tag{6.11a}\\
\omega_{1 d} & =k \omega_{3} \sin t  \tag{6.11b}\\
\delta & =\left(4+4 c_{3} k-c_{3} k \sin (2 t)\right) c_{3} \omega_{3} \tag{6.11c}
\end{align*}
$$

and $\dot{\omega}_{1 d}, \dot{\omega}_{2 d}$ are the derivatives of $\omega_{1 d}, \omega_{2 d}$ along the solutions of the closed loop system.
Proof. To prove Proposition 6.2 a similar approach as step IV and step V in Mazenc et al. (2002) will be used.

Consider the subsystem (6.9c) with $\omega_{1 d}$ and $\omega_{2 d}$ as virtual inputs:

$$
\begin{equation*}
\dot{\omega}_{3}=c_{3} \omega_{1 d} \omega_{2 d} \tag{6.12}
\end{equation*}
$$

Insertion of the velocity controllers in (6.11) gives

$$
\begin{equation*}
\dot{\omega}_{3}=-c_{3} k \frac{\omega_{3}^{3} \sin ^{2} t}{\beta+\omega_{3}^{2}} \tag{6.13}
\end{equation*}
$$

Consider the Lyapunov function

$$
\begin{equation*}
V_{1}=\omega_{3}^{2} \tag{6.14}
\end{equation*}
$$

Differentiation along the trajectories of (6.12) gives

$$
\begin{align*}
\dot{V}_{1} & =2 \omega_{3} \dot{\omega}_{3}  \tag{6.15}\\
& =-2 c_{3} k \frac{\omega_{3}^{4} \sin ^{2} t}{\beta+\omega_{3}^{2}}  \tag{6.16}\\
& \leq-c_{3} k \omega_{3}^{2} \sin ^{2} t  \tag{6.17}\\
& \leq-p(t) W\left(V_{1}\right) \leq 0 \tag{6.18}
\end{align*}
$$

where

$$
\begin{equation*}
p(t)=\sin ^{2} t \geq 0, \quad W\left(V_{1}\right)=c_{3} k \omega_{3}^{2}=c_{3} k V_{1}\left(\omega_{3}\right) \geq 0 \tag{6.19}
\end{equation*}
$$

Unfortunately, $\dot{V}_{1}$ is negative semi-definite. However, since $\dot{V}_{1}$ satisfies Assumptions A. 1 and A.2, Theorem A. 3 states that there exist a strict Lyapunov function for the system (6.12):

$$
\begin{equation*}
U=\Gamma\left(V_{1}\right)+P(t) \lambda\left(V_{1}\right) \tag{6.20}
\end{equation*}
$$

where $\Gamma(\cdot), \lambda(\cdot)$ are of class $\mathcal{K}_{\infty}$ and $P(t)$ is defined in (A.8). Since $p(t)=\sin ^{2} t$, with period $T=\pi, P(t)$ can be written as

$$
\begin{align*}
P(t) & =-t \int_{0}^{\pi} \sin ^{2} s d s+\pi \int_{0}^{t} \sin ^{2} s d s  \tag{6.21}\\
& =-\frac{\pi}{2} \sin (2 t) \tag{6.22}
\end{align*}
$$

To simplify the design we consider a slightly different function than (6.20):

$$
\begin{equation*}
V_{2}=2 V+\Gamma\left(V_{1}\right)+P_{2}(t) \lambda\left(V_{1}\right) \tag{6.23}
\end{equation*}
$$

Let $\lambda(V)=W(V)$ and $P_{2}(t)=\frac{1}{2} \sin (2 t) . \Gamma(V)$ is to be chosen later.

$$
\begin{align*}
V_{2} & =2 V_{1}+\Gamma\left(V_{1}\right)-\frac{1}{2} \sin (2 t) W\left(V_{1}\right)  \tag{6.24}\\
\dot{V}_{2} & \leq-2 \sin ^{2}(t) W(V)+\Gamma^{\prime}\left(V_{1}\right) \dot{V}_{1}-\cos (2 t) W\left(V_{1}\right)-\frac{1}{2} \sin (2 t) W^{\prime}\left(V_{1}\right) \dot{V}_{1}  \tag{6.25}\\
& =-\left(2 \sin ^{2} t+\cos (2 t)\right) W\left(V_{1}\right)+\left(\Gamma^{\prime}\left(V_{1}\right)-\frac{1}{2} \sin (2 t) W^{\prime}\left(V_{1}\right)\right) \dot{V}_{1}  \tag{6.26}\\
& \leq\left(\Gamma^{\prime}\left(V_{1}\right)-\frac{1}{2} \sin (2 t) W^{\prime}(V)\right) \dot{V}_{1}-W\left(V_{1}\right) \tag{6.27}
\end{align*}
$$

where we have used that $\cos (2 t)=\cos ^{2} t-\sin ^{2} t$. Furthermore $W^{\prime}\left(V_{1}\right)=c_{3} k$. Choosing $\Gamma^{\prime}\left(V_{1}\right)=2 c_{3} k$ yields

$$
\begin{align*}
\dot{V}_{2} & \leq \frac{3}{2} c_{3} k \dot{V}_{1}-W\left(V_{1}\right)  \tag{6.28}\\
& \leq-W\left(V_{1}\right)<0, \quad \forall \omega_{3} \neq 0 \tag{6.29}
\end{align*}
$$

since $|-\sin (2 t)| \leq 1$. With $\Gamma(V)=2 c_{3} k V=2 W(V)$ and $\lambda\left(V_{1}\right)=W\left(V_{1}\right),(6.23)$ is

$$
\begin{align*}
V_{2} & =2 V_{1}+2 W\left(V_{1}\right)-\frac{1}{2} \sin (2 t) W\left(V_{1}\right) \\
& =\left(2+2 c_{3} k\right) V_{1}-\frac{1}{2} \sin (2 t) c_{3} k V_{1} \tag{6.30}
\end{align*}
$$

and satisfies

$$
\begin{equation*}
2 V_{1} \leq V_{2} \leq\left(2+\frac{5}{2} c_{3} k\right) V_{1} \tag{6.31}
\end{equation*}
$$

All the conditions of Theorem A. 1 are satisfied, hence the subsystem (6.12) is globally uniformly asymptotically stable.

The derivative of the Lyapunov function $V_{2}$ along the trajectories of the complete system (6.9) satisfies

$$
\begin{equation*}
\dot{V}_{2}=\frac{\partial V_{2}}{\partial \omega_{3}} \dot{\omega}_{3}+\frac{\partial V_{2}}{\partial t} \tag{6.32}
\end{equation*}
$$

By writing $\dot{\omega}_{3}$ as

$$
\begin{equation*}
\dot{\omega}_{3}=c_{3}\left(\omega_{1} \omega_{2}+\omega_{1 d} \omega_{2 d}-\omega_{1 d} \omega_{2 d}\right) \tag{6.33}
\end{equation*}
$$

$\dot{V}_{2}$ can be written as

$$
\begin{equation*}
\dot{V}_{2}=\frac{\partial V_{2}}{\partial \omega_{3}} c_{3} \omega_{1 d} \omega_{2 d}+\frac{\partial V_{2}}{\partial \omega_{3}} c_{3}\left(\omega_{1} \omega_{2}-\omega_{1 d} \omega_{2 d}\right)+\frac{\partial V_{2}}{\partial t} . \tag{6.34}
\end{equation*}
$$

with

$$
\begin{equation*}
\frac{\partial V_{2}}{\partial \omega_{3}}=\left(4+4 c_{3} k-c_{3} k \sin (2 t)\right) \omega_{3}, \quad \frac{\partial V_{2}}{\partial t}=-\cos (2 t) c_{3} k V_{1} \tag{6.35}
\end{equation*}
$$

Using (6.28) it's seen that $\dot{V}_{2}$ satisfies

$$
\begin{equation*}
\dot{V}_{2} \leq-W\left(V_{1}\right)+\frac{\partial V_{2}}{\partial \omega_{3}} c_{3}\left(\omega_{1} \omega_{2}-\omega_{1 d} \omega_{2 d}\right) \tag{6.36}
\end{equation*}
$$

The next step is to find a control law that stabilizes the whole system (6.9). Consider the function

$$
\begin{equation*}
V_{3}=V_{2}+\frac{1}{2}\left(\omega_{1}-\omega_{1 d}\right)^{2}+\frac{1}{2}\left(\omega_{2}-\omega_{2 d}\right)^{2} \tag{6.37}
\end{equation*}
$$

Differentiation along the trajectories of (6.9) yields

$$
\begin{align*}
\dot{V}_{3}= & \dot{V}_{2}+\left(\omega_{1}-\omega_{1 d}\right)\left(\tau_{a}-\dot{\omega}_{1 d}\right)+\left(\omega_{2}-\omega_{2 d}\right)\left(\tau_{a}-\dot{\omega}_{2 d}\right)  \tag{6.38}\\
\leq & -W\left(V_{1}\right)+\frac{\partial V_{2}}{\partial \omega_{3}} c_{3}\left(\omega_{1} \omega_{2}-\omega_{1 d} \omega_{2 d}\right) \\
& +\left(\omega_{1}-\omega_{1 d}\right)\left(\tau_{a}-\dot{\omega}_{1 d}\right)+\left(\omega_{2}-\omega_{2 d}\right)\left(\tau_{a}-\dot{\omega}_{2 d}\right)  \tag{6.39}\\
\leq & -W\left(V_{1}\right)+\frac{\partial V_{2}}{\partial \omega_{3}} c_{3}\left(\left(\omega_{1}-\omega_{1 d}\right) \omega_{2}+\left(\omega_{2}-\omega_{2 d}\right) \omega_{1 d}\right) \\
& +\left(\omega_{1}-\omega_{1 d}\right)\left(\tau_{a}-\dot{\omega}_{1 d}\right)+\left(\omega_{2}-\omega_{2 d}\right)\left(\tau_{a}-\dot{\omega}_{2 d}\right) \tag{6.40}
\end{align*}
$$

Insertion of the controllers in (6.10) gives

$$
\begin{equation*}
\dot{V}_{3} \leq-W\left(V_{1}\right)-k_{a}\left(\omega_{1}-\omega_{1 d}\right)^{2}-k_{b}\left(\omega_{1}-\omega_{1 d}\right)^{2}<0 \quad \forall \omega_{1}, \omega_{2}, \omega_{3} \neq 0 \tag{6.41}
\end{equation*}
$$

The function $\dot{V}_{3}$ is negative definite and $V_{3}$ is radially unbounded. It then follows from Theorem A. 1 that the system (6.9) is globally uniformly asymptotically stable in closed-loop with the control laws in (6.10).

Remark 6.2.1. The controllers (6.10) are unnecessary complicated. Because of the strict Lyapunov function $V_{2}$ it is possible to use robust backstepping techniques to find simpler feedback laws that achieves global uniformly asymptotic stability. Unfortunately this results in a more complicated stability analysis. A more simple and robust controller is:

$$
\begin{align*}
\tau_{a} & =-k_{a}\left(\omega_{1}-\omega_{1 d}\right)+\dot{\omega}_{1 d}  \tag{6.42a}\\
\tau_{b} & =-k_{b}\left(\omega_{2}-\omega_{2 d}\right)+\dot{\omega}_{2 d} \tag{6.42b}
\end{align*}
$$

Simulations indicate that even the controller

$$
\begin{align*}
\tau_{a} & =-k_{a}\left(\omega_{1}-\omega_{1 d}\right)  \tag{6.43a}\\
\tau_{b} & =-k_{b}\left(\omega_{2}-\omega_{2 d}\right) \tag{6.43b}
\end{align*}
$$

achieves GUAS.

### 6.2.1 Simulation with the angular velocity controller

The control laws (6.2) and (6.3) has been simulated with the initial conditions

$$
\left[\omega_{1}(0), \omega_{2}(0), \omega_{3}(0)\right]^{\mathrm{T}}=[0.5,-0.5,1.0]^{\mathrm{T}}
$$

and parameters

$$
c_{3}=0.2, k_{a}=k_{b}=4, k=3, \beta=0.001
$$

Figure 6.5 and 6.6 shows the time evolution of the angular velocities and torque controls. The angular velocities are quickly reduced but the actuator usage is initially quite high. This can be reduced by changing the parameters. The controller is less effective with small values of $c_{3}$

### 6.3 Partial attitude stabilization

It is tempting to combine the angular velocity and spin axis controllers to try to achieve partial attitude stabilization, i.e, stabilize $\mathrm{w}_{1}, \mathrm{w}_{2}$ and $\omega_{3}$ to zero. From a practical point of view this can be just as important as full attitude stabilization. One example is a space telescope where the camera must be pointed in a specific direction and the spin rate about the telescope axis must be zero. The value of $z$ is not important as long as it is constant. The solution of the partial attittude stabilization problem is also a very important step towards full attitude stabilization.

The following reduced model with $\omega_{1}$ and $\omega_{2}$ as virtual inputs is considered:

$$
\begin{align*}
\dot{\mathrm{w}}_{1} & =\omega_{3} \mathrm{w}_{2}+\omega_{2} \mathrm{w}_{1} \mathrm{w}_{2}+\frac{\omega_{1}}{2}\left(1+\mathrm{w}_{1}^{2}-\mathrm{w}_{2}^{2}\right)  \tag{6.44a}\\
\dot{\mathrm{w}}_{2} & =-\omega_{3} \mathrm{w}_{1}+\omega_{1} \mathrm{w}_{1} \mathrm{w}_{2}+\frac{\omega_{2}}{2}\left(1+\mathrm{w}_{2}^{2}-\mathrm{w}_{1}^{2}\right)  \tag{6.44b}\\
\dot{\omega}_{3} & =c_{3} \omega_{1} \omega_{2} \tag{6.44c}
\end{align*}
$$

Proposition 6.3. The controllers

$$
\begin{align*}
& \omega_{1}=-k_{1} \mathrm{w}_{1}-\frac{\omega_{3}^{2} \sin t}{\left(\beta+\omega_{3}^{2}\right) \alpha(\mathrm{w})}  \tag{6.45a}\\
& \omega_{2}=-k_{2} \mathrm{w}_{2}+\frac{k_{3} \omega_{3} \sin t}{\alpha(\mathrm{w})} \tag{6.45b}
\end{align*}
$$

with

$$
\begin{equation*}
\alpha(\mathrm{w})=\sqrt{1+\mathrm{w}_{1}^{2}+\mathrm{w}_{2}^{2}}, \quad 0<\beta \ll 1 \tag{6.46}
\end{equation*}
$$

locally asymptotically stabilize the system (6.44)
Unfortunately a proof is not available at the time of writing. Extensive simulations indicate that Proposition 6.3 is true even for extreme initial values. The idea behind the controllers in (6.45) is that the first term stabilizes $\mathrm{w}_{1}$ and $\mathrm{w}_{2}$ to a neighbourhood about the origin while the second term stabilizes $\omega_{3}$. The intention of the term $\alpha(\mathrm{w})$ is to limit $\omega_{3}$ when $\mathrm{w}_{1}$ and $\mathrm{w}_{2}$ are large.

Many attempts have been made to find a continous controller that renders the origin of (6.45) globally uniformly asymptotically stable. The focus has been on smooth time-varying


Figure 6.5: The time evolution of the angular velocities $\omega_{1}(-), \omega_{2}(--), \omega_{3}(-\cdot)$ when using the angular velocity controller.


Figure 6.6: The time evolution of the torque controls $\tau_{a}(-), \tau_{b}(--)$ when using the angular velocity controller.
periodic control laws and to find Lyapunov functions that satisfies Theorem A.3. One simple candidate is

$$
\begin{equation*}
V=\ln \left(1+\mathrm{w}_{1}+\mathrm{w}_{2}\right)+\omega_{3}^{2} \tag{6.47}
\end{equation*}
$$

However, at the time of writing, it has not been shown that by using controllers with a similar structure as (6.45), $\dot{V}$ can be written as

$$
\begin{equation*}
\dot{V}=-p(t) W(V) \leq 0 \tag{6.48}
\end{equation*}
$$

### 6.3.1 Simulation with the partial attitude stabilization controller

The controllers in (6.45) have been simulated with the initial conditions

$$
\left[\mathrm{w}_{1}(0), \mathrm{w}_{2}(0), z(0), \omega_{3}(0)\right]^{\mathrm{T}}=\left[-1,1,-\frac{\pi}{2}, 1.5\right]^{\mathrm{T}}
$$

and parameters

$$
c_{3}=0.2, k_{1}=k_{2}=0.9, k_{3}=5, \beta=0.001
$$

The results are shown in Figure 6.7-6.8. Note that $\omega_{1}$ and $\omega_{2}$ are considered as actuators.

### 6.4 Attitude stabilization

Our ultimate goal is to find a controller that globally stabilizes the complete system

$$
\begin{align*}
\dot{\mathrm{w}}_{1} & =\omega_{3} \mathrm{w}_{2}+\omega_{2} \mathrm{w}_{1} \mathrm{w}_{2}+\frac{\omega_{1}}{2}\left(1+\mathrm{w}_{1}^{2}-\mathrm{w}_{2}^{2}\right)  \tag{6.49a}\\
\dot{\mathrm{w}}_{2} & =-\omega_{3} \mathrm{w}_{1}+\omega_{1} \mathrm{w}_{1} \mathrm{w}_{2}+\frac{\omega_{2}}{2}\left(1+\mathrm{w}_{2}^{2}-\mathrm{w}_{1}^{2}\right)  \tag{6.49b}\\
\dot{z} & =\omega_{3}-\omega_{1} \mathrm{w}_{2}+\omega_{2} \mathrm{w}_{1}  \tag{6.49c}\\
\dot{\omega}_{1} & =\tau_{a}  \tag{6.49d}\\
\dot{\omega}_{2} & =\tau_{b}  \tag{6.49e}\\
\dot{\omega}_{3} & =c_{3} \omega_{1} \omega_{2} \tag{6.49f}
\end{align*}
$$

As far as the Author know, at the time of writing there exist no controllers that globally stabilizes the attitude of an underactuated nonsymmetric rigid spacecraft. Global exponential convergence was indicated in Godhavn and Egeland (1995), but no proof was given. Possible candidates are suggested in Tsiotras and Doumtchenko (2000) but no formal proof is available.

Countless attempts have been made to find such control laws and several strategies have been tried during the work of this thesis. One promising strategy is to let $\mathrm{w}_{1}$ and $\mathrm{w}_{2}$ converge to a time varying manifold by introducing the change of variables

$$
\begin{equation*}
W_{1}=\mathrm{w}_{1}-g_{1}\left(z, \omega_{3}\right) h_{1}(t), \quad W_{2}=\mathrm{w}_{2}-g_{2}\left(z, \omega_{3}\right) h_{2}(t) \tag{6.50}
\end{equation*}
$$

where $h_{1}, h_{2}$ are time varying periodic functions. A similar approach was used in Morin et al. (1995) and Walsh et al. (1994) to determine smooth time-varying periodic control laws that locally renders the origin of the undearctuated spacecraft asymptotically stable. Both used averaging and center manifold theory to prove stability. A particular feature of Walsh et al. (1994) is a novel choice of Listing ${ }^{1}$ coordinates to describe the attitude. The Listing coordinates are quite similar to the $(\mathrm{w}, z)$-paramterization.

[^4]

Figure 6.7: The time evolution of the angular velocities $\omega_{1}(-), \omega_{2}(--)$ and $\omega_{3}$ (lower plot) when using the partial attitude stabilization controller.


Figure 6.8: The time evolution of the attitude when using the partial attitude stabilization controller. Upper plot shows $\mathrm{w}_{1}(-), \mathrm{w}_{2}(--)$.

The disadvantage of introducing a time varying change of variables is that the elegant symmetric structure of the ( $\mathrm{w}, z$ )-parameterization is destroyed, and extra terms are introduced. This makes it difficult to find a suitable Lyapunov function to prove global stability for the system. The same problem is observed in Morin et al. (1995) where the controllers derived have many terms and corresponding Lyapunov functions only locally stable.

### 6.4.1 Time-varying exponential stabililization

In Morin and Samson (1997) a time-varying periodic controller was proposed that locally exponentially stabilizes the attitude of an underactuated rigid spacecraft. The proposed controller uses Rodrigues parameters which are singular at $\theta= \pm \pi$. In this section the result is extended to the ( $\mathrm{w}, z$ )-parameterization, thereby avoiding the singular condition and extending the convergence range.

The following proposition is based on Theorem 1 in Morin and Samson (1997), with some changes in notation. See Appendix A for more details.

Proposition 6.4. Consider the functions

$$
\begin{align*}
& \omega_{1 d}=-k_{1} \mathrm{w}_{1}-\rho\left(\mathbf{x}, \omega_{3}\right) \sin (t / \epsilon)  \tag{6.51}\\
& \omega_{2 d}=-k_{2} \mathrm{w}_{2}+\frac{1}{\rho\left(\mathbf{x}, \omega_{3}\right)}\left(z+\omega_{3}\right) \sin (t / \epsilon) \tag{6.52}
\end{align*}
$$

with $\mathbf{x}=\left[\mathrm{w}_{1}, \mathrm{w}_{2}, z\right]^{T}$ and $\rho$, of class $\mathcal{C}^{1}$ on $\mathbb{R}^{4}-\{\mathbf{0}\}$, a homogeneous norm associated with the dilation

$$
\begin{equation*}
\delta_{\lambda}^{r}(\mathbf{x})=\left(\lambda \mathrm{w}_{1}, \lambda \mathrm{w}_{2}, \lambda^{2} z, \lambda^{2} \omega_{3}, t\right) \tag{6.53}
\end{equation*}
$$

and the following time-varying continous feedback:

$$
\begin{align*}
\tau_{a} & =k_{3}\left(\omega_{1}-\omega_{1 d}\right)  \tag{6.54a}\\
\tau_{b} & =k_{4}\left(\omega_{2}-\omega_{2 d}\right) . \tag{6.54b}
\end{align*}
$$

Then, for any positive parameters $k_{1}$ and $k_{2}$, there exists $\epsilon_{0}>0$ such that for any $\epsilon \in\left(0, \epsilon_{0}\right]$ and large enough parameters $k_{3}>0$ and $k_{4}>0$, the feedback (6.54) locally asymptotically and exponentially stabilizes the origin of (6.49).

A proof of Proposition 6.4 can be found in Morin and Samson (1997). Unfortunately the proof is based on a attitude parameterization using Rodrigues parameters, however, it is shown below that the proof can be applied directly to the $(\mathrm{w}, z)$-parameterization as well.

The system can be written as the perturbed system

$$
\left[\begin{array}{c}
\dot{\mathbf{x}}  \tag{6.55}\\
\dot{\boldsymbol{\omega}}
\end{array}\right]=\mathbf{f}(\mathbf{x}, \boldsymbol{\omega}, t)+\mathbf{h}(\mathbf{x}, \boldsymbol{\omega}, t)
$$

where

$$
\mathbf{x}=\left[\mathrm{w}_{1}, \mathrm{w}_{2}, z\right]^{\mathrm{T}}, \quad \boldsymbol{\omega}=\left[\omega_{1}, \omega_{2}, \omega_{3}\right]^{\mathrm{T}}
$$

and

$$
\mathbf{f}(\mathbf{x}, \boldsymbol{\omega}, t)=\left[\begin{array}{c}
\frac{1}{2} \omega_{1}  \tag{6.56}\\
\frac{1}{2} \omega_{1} \\
\omega_{3}+\omega_{2} \mathrm{w}_{1}-\omega_{1} \mathrm{w}_{2} \\
\tau_{a}(\mathbf{x}, \boldsymbol{\omega}, t) \\
\tau_{a}(\mathbf{x}, \boldsymbol{\omega}, t) \\
c_{3} \omega_{1} \omega_{2}
\end{array}\right], \quad \mathbf{h}(\mathbf{x}, \boldsymbol{\omega}, t)=\left[\begin{array}{c}
\mathrm{w}_{1} \mathrm{w}_{2} \omega_{2}+\mathrm{w}_{2} \omega_{3}+\frac{1}{2} \omega_{1}\left(\mathrm{w}_{1}^{2}+\mathrm{w}_{2}^{2}\right) \\
\mathrm{w}_{1} \mathrm{w}_{2} \omega_{1}-\mathrm{w}_{1} \omega_{3}+\frac{1}{2} \omega_{2}\left(\mathrm{w}_{2}^{2}-\mathrm{w}_{1}^{2}\right) \\
0 \\
0 \\
0 \\
0
\end{array}\right]
$$

When using Rodrigues parameters $\mathbf{f}(\mathbf{x}, \boldsymbol{\omega}, t)$ and $\mathbf{g}(\mathbf{x}, \boldsymbol{\omega}, t)$ are

$$
\mathbf{f}_{q}(\cdot)=\left[\begin{array}{c}
\frac{1}{2} \omega_{1}  \tag{6.57}\\
\frac{1}{2} \omega_{1} \\
\frac{1}{2}\left(\omega_{3}+\omega_{2} q_{1}-\omega_{1} q_{2}\right) \\
\tau_{a}(\mathbf{x}, \boldsymbol{\omega}, t) \\
\tau_{a}(\mathbf{x}, \boldsymbol{\omega}, t) \\
c_{3} \omega_{1} \omega_{2}
\end{array}\right], \quad \mathbf{h}_{q}(\cdot)=\left[\begin{array}{c}
\frac{1}{2}\left(q_{1}^{2} \omega_{1}+\left(q_{1} q_{2}-q_{3}\right) \omega_{2}+\left(q_{1} q_{3}+q_{2}\right) \omega_{3}\right) \\
\frac{1}{2}\left(\left(q_{1} q_{2}+q_{3}\right) \omega_{1}+q_{2}^{2} \omega_{2}+\left(q_{2} q_{3}-q_{1}\right) \omega_{3}\right) \\
\frac{1}{2}\left(q_{3} q_{1} \omega_{1}+q_{3} q_{2} \omega_{2}+q_{3}^{2} \omega_{3}\right) \\
0 \\
0 \\
0
\end{array}\right]
$$

with $\mathbf{x}=\left[q_{1}, q_{2}, q_{3}\right]^{\mathrm{T}}$. The functions $\mathbf{f}(\cdot)$ and $\mathbf{f}_{q}(\cdot)$ are identical in sturcture, except for the constant $\frac{1}{2}$, i.e. $f_{3}(\cdot)=2 f_{q 3}(\cdot)$. Proposition A. 1 states that $\mathbf{h}(\mathbf{x}, t)$ must be a T-periodic function such that the corresponding vector field $\mathbf{h}$ is a sum of homogeneous vector fields of degree strictly positive with respect to the dilation

$$
\begin{equation*}
\delta_{e}^{r}=\left(\lambda \mathrm{w}_{1}, \lambda \mathrm{w}_{2}, \lambda^{2} z, \lambda \omega_{1}, \lambda \omega_{2}, \lambda^{2} \omega_{3}, t\right) \tag{6.58}
\end{equation*}
$$

A direct calculation yields that

$$
\left[\begin{array}{c}
h_{1}\left(\delta_{e}^{r}\right)  \tag{6.59}\\
h_{2}\left(\delta_{e}^{r}\right) \\
h_{3}\left(\delta_{e}^{r}\right) \\
\vdots
\end{array}\right]=\left[\begin{array}{c}
\lambda^{1+2} h_{1}(\mathbf{x}, \boldsymbol{\omega}, t) \\
\lambda^{1+2} h_{2}(\mathbf{x}, \boldsymbol{\omega}, t) \\
\lambda^{2+2} 0 \\
\vdots
\end{array}\right]
$$

and therefore $\mathbf{g}(\mathbf{x}, t)$ is of degree 2 with respect to $\delta_{e}^{r}$. The function $\mathbf{h}_{q}(\cdot)$ is also of degree 2 with respect to

$$
\begin{equation*}
\delta_{q}^{r}=\left(\lambda q_{1}, \lambda q_{2}, \lambda^{2} q_{3}, \lambda \omega_{1}, \lambda \omega_{2}, \lambda^{2} \omega_{3}, t\right) \tag{6.60}
\end{equation*}
$$

Since the two systems have the same properties it then follows from Theorem 1 in Morin and Samson (1997) that Proposition 6.4 is correct.

### 6.4.2 Simulation with the time-varying exponential controller

The control laws (6.2) and (6.3) have been simulated with initial conditions

$$
\left[\mathrm{w}_{1}(0), \mathrm{w}_{2}(0), z(0), \omega_{1}(0), \omega_{2}(0), \omega_{3}(0)\right]^{\mathrm{T}}=\left[2,-2, \frac{\pi}{2}, 1,-1,1\right]^{\mathrm{T}}
$$

and parameters

$$
c_{3}=0.5, k_{1}=k_{2}=1, k_{3}=k_{4}=6, \epsilon=\frac{1}{3}
$$

Figure 6.9-6.11 shows the results of the simulations.
Remark 6.4.1. Note that the time responses are quite oscillary, except for $\omega_{3}$ and $z$. The high frequent motion can be a serious problems if the spacecraft consists of flexible parts. An advantage of the controllers derived with the tools from Mazenc et al. (2002) is that they work for low frequencies.


Figure 6.9: The time evolution of the attitude using when using the exponential stable controller


Figure 6.10: The time evolution of the angular velocities when using the exponential stable controller.


Figure 6.11: The time evolution of the torque controls when using the exponential stable controller

## Chapter 7 <br> Conclusions

In this thesis the attitude stabilization of an underactuated rigid spacecraft has been studied. The spacecraft has been modelled as an ideal rigid body and the relatively new (w,z)parameterization has been used to represent the attitude due to its interesting and favorable properties compared with other minimal attitude representations. Several properties of the underactuated spacecraft have been presented, most important being the fact that it does not satisfy Brockett's necessary condition and therefore can't be stabilized using time-invariant continuous state feedback.

A result in Mazenc et al. (2002), solving the open problem of determining explicit expressions of smooth time-varying periodic state feedbacks which render the origin of an underactuated surface vessel globally uniformly asymptotically stable, has been studied and the possibility of extending the result to the underactuated spacecraft attitude stabilization problem has been investigated. It has been shown that direct application is not possible because of the lack of damping in the spacecraft dynamics and the spacecraft model is considerably more complicated. than the surface vessel model. However the article provides several useful tools and methods.

Insight about the attitude stabilization problem has been gained by solving the subproblems of spin-axis stabilization and angular velocity stabilization. The angular velocity controller has shown that some of the results of Mazenc et al. (2002) can be applied to the spacecraft stabilization problem. An attempt to combine the two controllers to achieve partial attitude stabilization has been made. Extensive simulations has indicated that such a controller is stable, but no proof of stability is available at the time of writing.

Several attempts to find continuous controllers that achieve global asymptotic attitude stabilization have failed. However, with more time it is probable that it is possible to find such controllers using the tools in Mazenc et al. (2002)

Finally a time-variant periodic local exponential stable controller has been demonstrated by extending the results of Morin and Samson (1997) to the (w,z)-parameterization.

### 7.1 Recommendations for further work

After finishing this thesis there is definitely more to do on the subject of attitude stabilization of underactuated spacecrafts. First of all it feels that the interesting work has just started and it is a bit sad to end a work in progress. Some recommendations for future work are:

- More attempts should be made to find globally controllers. A considerable part of the
time has been used to understand the problem and to study litterature.
- How is the problem affected by introducing a gravity gradient?
- Reaction wheels have not been considered in this work. Results in Crouch (1984) states that attitude stabilization is not possible with momentum exchange devices. What is the best that can be achieved with such devices?
- Tracking and robust attitude control of underactuated spacecrafts is still open problems.


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## Appendix A

## Theory

## A. 1 Lyapunov stability

First some definitions.
Definition A.1. A real valued function $\gamma(s)$ is of class $\mathcal{K}_{\infty}$ if it is continuous, strictly increasing and satisfies

$$
\gamma(0)=0, \quad \lim _{s \rightarrow+\infty} \gamma(s)=+\infty
$$

Consider the nonautonomous system

$$
\begin{equation*}
\dot{\mathbf{x}}=\mathbf{f}(\mathbf{x}, t) \tag{A.1}
\end{equation*}
$$

where $f:[0, \infty) \times D \rightarrow \mathbb{R}^{n}$.
Theorem A.1. (Khalil (1996), Theorem 3.8) Let $\mathbf{x}=0$ be an equilibrium for (A.1) and $D \subset$ $\mathbb{R}^{n}$ be a domain containing $\mathbf{x}=\mathbf{0}$. Let $V:[0, \infty) \times D \rightarrow \mathbb{R}$ be a continuously differentiable function such that

$$
\begin{gather*}
W_{1}(x) \leq V(\mathbf{x}, t) \leq W_{2}(\mathbf{x})  \tag{A.2a}\\
\frac{\partial V(\mathbf{x}, t)}{\partial t}+\frac{\partial V(, \mathbf{x})}{\partial t} f(\mathbf{x}, t) \leq-W_{3}(\mathbf{x}) \tag{A.2b}
\end{gather*}
$$

$\forall t \geq t_{0}, \forall \mathbf{x} \in D$ where $W_{1}(\mathbf{x}), W_{2}(\mathbf{x})$ and $W_{3}(\mathbf{x})$ are continuous positive definite functions on $D$. Then, $\mathbf{x}=\mathbf{0}$ is locally uniformly asymptotically stable.

Corollary A.1. Suppose that all the assumption of Theorem A. 1 are satisfied globally (for all $\mathbf{x} \in \mathbb{R}^{n}$ ) and $W_{1}(\mathbf{x})$ is radially unbounded. Then, $\mathbf{x}=\mathbf{0}$ is globally uniformly asymptotically stable.

Definition A.2. A Lyapunov function satisfying (A.2a) and (A.2b) is called a strict Lyapunov function

In Brockett (1985) a necessary condition for stabilizability by continuous time-invariant feedback was presented. It is often referred to as Brockett's necessary condition or Brockett's theorem. It was shown to hold for $\mathcal{C}^{1}$ time-invariant state feedback, and in Zabczyk (1989) it was shown to hold for continuous time-invariant state feedback also. It can be formulated as follows (Aneke, 2003):

Theorem A.2. Assume that there exists a continuous time-invariant state feedback $\mathbf{u}: \mathbb{R}^{n} \rightarrow$ $\mathbb{R}^{m}$, that renders the origin of

$$
\begin{equation*}
\dot{\mathbf{x}}=\mathbf{f}(\mathbf{x}, \mathbf{u}) \tag{A.3}
\end{equation*}
$$

with $\mathbf{x} \in \mathbb{R}^{n}$ and $\mathbf{u} \in \mathbb{R}^{m}$, asymptotically stable. Then the function $\mathbf{f}: \mathbb{R}^{n} \times \mathbb{R}^{m} \rightarrow \mathbb{R}^{n}$ is locally surjectible, i.e., the function $\mathbf{f}$ maps an arbitrary neighborhood of $(\mathbf{0}, \mathbf{0}) \in \mathbb{R}^{n} \times \mathbb{R}^{m}$ onto a neighborhood of $\mathbf{0}$ in $\mathbb{R}^{n}$.

## A. 2 Strict Lyapunov functions for time-varying systems

Consider the time varying system

$$
\begin{equation*}
\dot{\mathbf{x}}=\mathbf{f}(\mathbf{x}, t) \tag{A.4}
\end{equation*}
$$

with $\mathbf{x} \in \mathbb{R}^{n}$ and $\mathbf{f}(\mathbf{x}, t)$ is a nonlinear function periodic in time of period $T>0$.
Assumption A.1. A Lyapunov function $V(\mathbf{x}, t)$, periodic in time and of period $T>0$, a positive definite function $W(\mathbf{x})$ and a nonnegative function $p(t)$, periodic and of period $T$, such that

$$
\begin{equation*}
\frac{\partial V}{\partial t}(\mathbf{x}, t)+\frac{\partial V}{\partial x}(\mathbf{x}, t) \mathbf{f}(\mathbf{x}, t) \leq-p(t) W(\mathbf{x}) \tag{A.5}
\end{equation*}
$$

and two functions $\alpha_{i}(\cdot), i=1,2$ of class $\mathcal{K}_{\infty}$ such that

$$
\begin{equation*}
\alpha_{1}(|\mathbf{x}|) \leq V(\mathbf{x}, t) \leq \alpha_{2}(|\mathbf{x}|) \tag{A.6}
\end{equation*}
$$

are known.
Assumption A.2. The constant $\int_{0}^{T} p(s) d s$ is strictly positive.
Theorem A.3. (Mazenc, 2003) If Assumptions A. 1 and A. 2 are satisfied by the system (A.4), one can determine the explicit expressions of a continuously differentiable function $\Gamma(\cdot)$ of class $\mathcal{K}_{\infty}$ and of a positive definite function $\lambda(\cdot)$ continuously differentiable, with a positive first derivate, such that the function

$$
\begin{equation*}
U(\mathbf{x}, t)=\Gamma(V(\mathbf{x}, t))+P(t) \lambda(V(\mathbf{x}, t)) \tag{A.7}
\end{equation*}
$$

with

$$
\begin{equation*}
P(t)=-t \int_{0}^{T} p(s) d s+T \int_{0}^{t} p(s) d s \tag{A.8}
\end{equation*}
$$

is a strict Lyapunov function for system (A.4).

## A. 3 Dilations and homogeneity

Some definitions about homogeneous systems are presented below. The definitions are based on Pettersen (1996) and Morin and Samson (1997).

For any $\lambda>0$ and any set of real parameters $r_{1}, \cdots, r_{n}>0$, a dilation operator $\delta_{\lambda}^{r}: \mathbb{R}^{n+1} \rightarrow$ $\mathbb{R}^{n+1}$ is defined by

$$
\begin{equation*}
\delta_{\lambda}^{r}\left(x_{1}, \ldots, x_{n}\right)=\left(\lambda^{r_{1}} x_{1}, \ldots, \lambda^{r_{n}} x_{n}, t\right) \tag{A.9}
\end{equation*}
$$

A homogeneous norm associated with the dilation $\delta_{\lambda}^{r}$ is

$$
\begin{equation*}
\rho_{p}^{r}(\mathbf{x})=\left(\sum_{i=1}^{n}\left|x_{i}\right|^{\frac{p}{r_{i}}}\right)^{\frac{1}{p}} \quad \text { with } \quad p>0 \tag{A.10}
\end{equation*}
$$

A continuous function $h: \mathbb{R}^{n} \times \mathbb{R} \rightarrow \mathbb{R}$ is said to be homogeneous of degree $\tau \geq 0$ with respect to the dilation $\delta_{\lambda}^{r}$ if

$$
\begin{equation*}
h\left(\delta_{\lambda}^{r}(\mathbf{x}, t)\right)=\lambda^{\tau}(\mathbf{x}, t) \quad \forall \lambda>0 \tag{A.11}
\end{equation*}
$$

A differential system $\dot{\mathbf{x}}=\mathbf{f}(\mathbf{x}, t)$ with $\mathbf{f}: \mathbb{R}^{n} \times \mathbb{R} \rightarrow \mathbb{R}^{n}$ continuous, is homogeneous of degree $\sigma \geq 0$ with respect to the dilation $\delta_{\lambda}^{r}$ if its $i$ th coordinate is a homogeneous function of degree $r_{i}+\sigma$, i.e.

$$
\begin{equation*}
f^{i}\left(\delta_{\lambda}^{r}(\mathbf{x}, t)\right)=\lambda^{r_{i}+\sigma} f^{i}(\mathbf{x}, t) \quad \forall \lambda>0 \quad i=1, \ldots, n \tag{A.12}
\end{equation*}
$$

Proposition A.1. (Morin and Samson, 1997) Consider the system

$$
\begin{equation*}
\dot{\mathbf{x}}=\mathbf{f}(\mathbf{x}, t) \tag{A.13}
\end{equation*}
$$

with $\mathbf{f}(\mathbf{x}, t): \mathbb{R}^{n} \times \mathbb{R} \rightarrow \mathbb{R}^{n}$ a T-periodic continuous function $(\mathbf{f}(\mathbf{x}, t+T)=\mathbf{f}(\mathbf{x}, t))$ and $\mathbf{f}(\mathbf{0}, t)=\mathbf{0}$. Assume that (A.13) is homogeneous of degree zero with respect to a dilation $\delta_{\lambda}^{r}(\mathbf{x}, t)$ and that the equilibrium point $\mathbf{x}=\mathbf{0}$ of this system is locally asymptotically stable. Then:

1. $\mathrm{x}=\mathbf{0}$ is globally exponentially stable in the sense that there exist two strictly positive constants $K$ and $\gamma$ such that along any solution of (A.13)

$$
\begin{equation*}
\rho_{p}^{r}(\mathbf{x}(t)) \leq K e^{-\gamma t} \rho_{p}^{r}(\mathbf{0}(t)) \tag{A.14}
\end{equation*}
$$

with $\rho_{p}^{r}(\mathbf{x})$ denoting a homogeneous norm associated with the dilation $\delta_{\lambda}^{r}(\mathbf{x}, t)$
2. the solution $\mathbf{x}=\mathbf{0}$ of the perturbed system

$$
\begin{equation*}
\dot{\mathbf{x}}=\mathbf{f}(\mathbf{x}, t)+\mathbf{h}(\mathbf{x}, t) \tag{A.15}
\end{equation*}
$$

is locally exponentially stable when $\mathbf{h}(\mathbf{x}, t): \mathbb{R}^{n} \times \mathbb{R} \rightarrow \mathbb{R}^{n}$ is a continuous $T$-periodic function such that the corresponding vector field $\mathbf{h}$ is a sum of homogeneous vector field of degree strictly positive with respect to $\delta_{\lambda}^{r}$

## Appendix B

## Newton-Euler equations for rigid bodies

Equations of motion for a rigid body can be derived by summing up the equations of motion for individual mass elements $d m$ with velocity $\vec{v}_{p}$. A rigid body $B$ with a mass element $d m$ is shown in Figure B.1. The point $c$ is the center of mass, while $o$ is the point where we want to express the equations of motion about. The material in this chapter is based on Egeland and Gravdahl (2002) and Fossen (2002).


Figure B.1: Rigid body with mass element $d m$.

## B. 1 Translational motion

The translational equation of motion with reference to a point $o$ can be written as

$$
\begin{equation*}
\vec{f}_{o}=m \vec{a}_{c} . \tag{B.1}
\end{equation*}
$$

From Figure B. 1 we have that $\vec{r}_{c}=\vec{r}_{o}+\vec{r}_{g}$, hence

$$
\begin{align*}
\vec{v}_{c} & =\vec{v}_{o}+\vec{\omega}_{i b} \times \vec{r}_{g},  \tag{B.2}\\
\vec{a}_{c} & =\vec{a}_{o}+\dot{\vec{\omega}}_{i b} \times \vec{r}_{g}+\vec{\omega}_{i b} \times\left(\vec{\omega}_{i b} \times \vec{r}_{g}\right), \tag{B.3}
\end{align*}
$$

where we have used that $\vec{r}_{g}$ is a constant in $b$.
Combining (B.1) and (B.3) gives the force equation with reference to the point $o$ :

$$
\begin{equation*}
\vec{f}_{o}=m\left(\vec{a}_{o}+\dot{\vec{\omega}}_{i b} \times \vec{r}_{g}+\vec{\omega}_{i b} \times\left(\vec{\omega}_{i b} \times \vec{r}_{g}\right)\right) \tag{B.4}
\end{equation*}
$$

The translational motion of a spacecraft can be controlled using thrusters. For a spacecraft in orbit the motion is governed by the laws of orbital mechanics. Such a law is the restricted two-body equation of motion:

$$
\begin{equation*}
\vec{a}=-\mu \frac{\vec{r}}{|r|^{3}} \tag{B.5}
\end{equation*}
$$

where $\vec{r}$ is the spacecraft's position and $\mu$ is the gravitational parameter for Earth. For more details see a textbook in orbital mechanics, for instance Prussing and Conway (1993).

## B. 2 Angular motion

The Newton-Euler equations are derived from Euler's First and Second Axioms:

$$
\begin{align*}
\vec{f}_{c} & =m \vec{a}_{c}  \tag{B.6}\\
\vec{\tau}_{c} & =\dot{\vec{h}}_{c}  \tag{B.7}\\
\vec{\tau}_{o} & =\vec{\tau}_{c}+\vec{r}_{g} \times \vec{f}_{c} \tag{B.8}
\end{align*}
$$

where the angular momentum about $c$ and $o$ are defined as

$$
\begin{align*}
\vec{h}_{c} & =\int_{B}\left(\vec{r} \times \vec{v}_{p}\right) d m  \tag{B.9}\\
\vec{h}_{o} & =\int_{B}\left(\overrightarrow{r_{d}} \times \vec{v}_{p}\right) d m \tag{B.10}
\end{align*}
$$

By using that $\vec{v}_{p}=\vec{v}_{o}+\vec{\omega}_{i b} \times \vec{r}_{d}$ and $\vec{r}_{d}=\vec{r}+\vec{r}_{g}$, (B.10) can be written as

$$
\begin{equation*}
\vec{h}_{o}=m \vec{r}_{g} \times \vec{v}_{o}+\int_{B} \vec{r}_{d} \times\left(\boldsymbol{\omega}_{i b} \times \vec{r}_{d}\right) d m \tag{B.11}
\end{equation*}
$$

To simplify (B.11) the inertia dyadic

$$
\begin{equation*}
I_{o}=\int_{B}-S^{2}\left(\vec{r}_{d}\right) d m \tag{B.12}
\end{equation*}
$$

is introduced. The angular momentum about $o$ can then be written as

$$
\begin{equation*}
\vec{h}_{o}=m \vec{r}_{g} \times \vec{v}_{o}+I_{o} \vec{\omega}_{i b} . \tag{B.13}
\end{equation*}
$$

An alternative expression can be found by writing $\vec{h}_{o}$ as

$$
\begin{align*}
\vec{h}_{o} & =\int_{B}\left(\vec{r}+\vec{r}_{g}\right) \times \vec{v}_{p} d m \\
& =\vec{h}_{c}+\int_{B}\left(\vec{r}_{g} \times \vec{v}_{p}\right) d m \\
& =\vec{h}_{c}+\vec{r}_{g} \times m \vec{r}_{c} \tag{B.14}
\end{align*}
$$

where we have used that $\vec{v}_{c} \equiv \frac{1}{m} \int_{B} \vec{v}_{p} d m$.
Time differentiation of $\vec{h}_{o}$ with respect to the inertial frame yields ${ }^{1}$

$$
\begin{equation*}
\dot{\vec{h}}_{o}=\vec{v}_{c} \times m \vec{v}_{o}+\vec{r}_{g} \times m \dot{\vec{v}}_{o}+\vec{M}_{o} \dot{\vec{\omega}}_{i b}+\vec{\omega}_{i b} \times\left(\vec{M}_{o} \vec{\omega}_{i b}\right) \tag{B.15}
\end{equation*}
$$

Equation (B.14) implies that

$$
\begin{equation*}
\dot{\vec{h}}_{o}=\dot{\vec{h}}_{c}+\vec{r}_{g} \times m \dot{\vec{v}}_{c}-\vec{v}_{o} \times m \vec{v}_{c} \tag{B.16}
\end{equation*}
$$

which combined with (B.15) gives

$$
\begin{equation*}
\dot{\vec{h}}_{c}=\vec{\tau}_{c}=\vec{r}_{g} \times m\left(\dot{\vec{v}}_{o}-\dot{\vec{v}}_{c}\right)+\vec{M}_{o} \dot{\vec{\omega}}_{i b}+\vec{\omega}_{i b} \times\left(\vec{M}_{o} \vec{\omega}_{i b}\right) \tag{B.17}
\end{equation*}
$$

Insertion of (B.17) in (B.8) and using (B.7) gives the angular equation of motion

$$
\begin{equation*}
\vec{\tau}_{o}=\vec{r}_{g} \times m \vec{a}_{o}+\vec{M}_{o} \dot{\vec{\omega}}_{i b}+\vec{\omega}_{i b} \times\left(\vec{M}_{o} \vec{\omega}_{i b}\right) \tag{B.18}
\end{equation*}
$$

## B. 3 Model summary

The equations (B.4) and (B.18) can be simplified by letting $o$ coincide with the center of mass $c$, meaning $\overrightarrow{r_{g}}=\overrightarrow{0}$ and $\vec{M}_{o}=\vec{M}_{c}$. The simplified equations are

$$
\begin{align*}
& \vec{f}=m \vec{a}  \tag{B.19a}\\
& \vec{\tau}=\vec{M} \dot{\vec{\omega}}_{i b}+\vec{\omega}_{i b} \times\left(\vec{M} \vec{\omega}_{i b}\right) \tag{B.19b}
\end{align*}
$$

where the subscript $c$ has been dropped for convenience.
Writing the equations of motion in coordinate form in the $b$ frame yields

$$
\begin{gather*}
m \dot{\mathbf{v}}^{b}=\mathbf{f}^{b}  \tag{B.20}\\
\mathbf{M} \dot{\boldsymbol{\omega}}_{i b}^{b}+\mathbf{S}\left(\boldsymbol{\omega}_{i b}^{b}\right) \mathbf{M} \boldsymbol{\omega}_{i b}^{b}=\boldsymbol{\tau}^{b} \tag{B.21}
\end{gather*}
$$

At a first glance the translational and angular motion seems decoupled. A closer inspection reveals that this is not the case. The reason is that disturbance torques, $\vec{\tau}_{d}$, and forces, $\vec{f}_{d}$ acting on a spacecraft are usually dependent of the spacecraft's position and attitude. However, for our purposes the translational and angular motion can be assumed decoupled.

[^5]
[^0]:    ${ }^{1}$ For the latest news about NCUBE, visit the official web-page: http://www.rocketrange.no/ncube/

[^1]:    ${ }^{1}$ An example from everyday life is the movement of for instance a cabinet or refridgerator by rocking it back and forth.

[^2]:    ${ }^{1}$ See Mazenc et al. (2002) for details.

[^3]:    ${ }^{2}$ See Definition A. 1

[^4]:    ${ }^{1}$ Listing coordinates are based on the works of Listing, a psychologist studying the movement of the eye. He noted that the eye moves in a way which minimally twists the optic nerve. Listing's law describes the subset of $S O(3)$ swept out by such an eye (Walsh et al., 1994)

[^5]:    ${ }^{1}$ For more details about the derivation refer to Egeland and Gravdahl (2002)

