Reachability-Based Approach for $\mathcal{L}_2$ Gain Approximation of Polynomial Systems

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Abstract— **This paper considers a computational mechanism to approximate a class of polynomial systems with a reduced order model. The approach is based on estimate of the reachability set and a finite gain $\mathcal{L}_2$ stability condition. The approach benefits from the use of sum of squares programming where the computation is rendered tractable.**

I. INTRODUCTION

There have been numerous procedures to reduce the number of states of nonlinear systems. Among the existing approaches, an approach for truncating nonlinear system is given by [8]. The method first computes a controllability function and an observability function from Hamilton-Jacobi equations. In general those functions are not entirely balanced. By seeking a coordinate transformation the original system is made balance to some extent of definition. A reduced order model is obtained by truncating the balanced system. The drawback of this method is on the computation of the controllability and observability functions which in most cases are very difficult.

One way to avoid this problem is introduced in [7] where the authors consider generalized controllability and observability functions which are obtained through Hamilton-Jacobi inequalities instead of Hamilton-Jacobi equalities. Despite the fact that the truncation scheme based on these generalized functions will not guarantee to give a stable reduced order model for a stable original system, the advantage of this approach is that it exploits the use of sum of squares programming [6] to compute the generalized functions and thus is amenable to computer solution in case the original system to be reduced has polynomial vector fields.

In attempting to utilize the power of sum of squares programming a heuristic approach is introduced in [9]. The approach computes a reduced model for polynomial system such that the error model satisfies a finite gain $\mathcal{L}_2$ stability condition. Though verification of this condition can be done through sum of squares programming, the method suffers from the coupling of the unknown storage function with the unknown structure of the reduced model which makes the computation untractable. To avoid this coupling of unknown variables, the generalized controllability and observability functions which satisfy the same type of Hamilton-Jacobi inequalities like in [7] are computed through sum of squares programming. Based on the generalized functions a storage function is constructed. The reduced model can then be computed such that the error model satisfies the finite gain $\mathcal{L}_2$ stability condition.

In this paper we try a different approach to decouple the unknown variables for verifying the finite gain $\mathcal{L}_2$ stability condition. Instead of constructing the storage function as the first step to avoid the coupling of the unknown variables, we construct partially the structure of the reduced model which is coupled with the storage function. The construction is based on an estimate of the reachability set of the system when the initial condition is set to the origin. In this case we seek the part of the system which is strongly reachable. This part of the system will be the state space of the reduced model while the output of the reduced model is obtained through sum of squares programming of relaxation of the finite gain $\mathcal{L}_2$ stability condition. The method is restricted to a certain class of polynomial systems.

II. PRELIMINARY DEVELOPMENTS

A. Notation

The following notation will be used throughout the paper. The set of polynomial in x with real coefficient is denoted by $\mathbb{R}[x]$. The set of matrices of size $n \times m$ whose entries are polynomial in x with real coefficient is denoted by $\mathbb{R}^{n\times m}[x]$. The superscript $'$ stands for matrix transposition. The notation I_p means the identity matrix of dimension $p \times p$. A scalar function $w(x)$ is said to be positive definite if $w(0) = 0$ and $w(x) > 0$ for all $x \neq 0$. The set of symmetric matrices in $\mathbb{R}^{n\times n}$ is denoted by S_n. The matrix inequality $W(x) \succeq 0 (\preceq 0)$ means that W is a positive (negative) semidefinite symmetric matrix for all x while $W(x) \succ 0 (\prec 0)$ means that W is a positive (negative) definite symmetric matrix for all x. The notation $\|\cdot\|$ means the Euclidean norm of the vector involved. $\mathcal{L}_2[0,T]$ means the vector space of function $\upsilon : [0,\infty) \to \mathbb{R}^q$ that satisfies

$$\int_0^T \|\upsilon(t)\|^2 dt < \infty.$$

A linear system $\hat{G}$ with realization $\{\hat{A}, \hat{B}, \hat{C}, \hat{D}\}$ can be written in a state space form

$$\dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}\hat{u},$$
$$\hat{y} = \hat{C}\hat{x} + \hat{D}\hat{u},$$

where $\hat{x}$ is the state, $\hat{u}$ is the input and $\hat{y}$ is the output. The ball in $\mathbb{R}^n$ is denoted by

$$B_r = \left\{ x \in \mathbb{R}^n | \ \|x\|^2 \leq r \right\}.$$

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B. $\mathcal{L}_2$ Gain Approximation

This paper is concerned with a nonlinear system

$$\dot{x} = f(x) + B(x)u, \qquad (1)$$
$$y = h(x) + D(x)u,$$

where $x = [x_1, \ldots, x_n]' \in \mathbb{R}^n$ is the state vector of the system, $u \in \mathbb{R}^{n_u}$ is the input to the system, $y \in \mathbb{R}^{n_y}$ is the output of the system and $f(x) \in \mathbb{R}^n[x]$, $h(x) \in \mathbb{R}^{n_y}[x]$, $B(x) \in \mathbb{R}^{n \times n_u}[x]$, $D(x) \in \mathbb{R}^{n_y \times n_u}[x]$ are polynomials in x, and therefore smooth. Throughout the paper we refer to such a system as polynomial system. We assume the following.

Assumption 1 *There exists $Q \succ 0$ such that*

$$x' f(x) \leq -x' Q x$$

for all $x \in \mathbb{R}^n$.

By the assumption the origin of the unforced system is globally asymptotically stable. For linear system $f(x) = Ax$ the assumption means that $A + A^{'}$ should be negative definite.

We consider a reduced order model

$$\dot{x}_r = f_r(x_r) + B_r(x_r)u, \qquad (2)$$
$$y_r = h_r(x_r) + D_r(x_r)u,$$

where $x_r = [x_{r1}, \ldots, x_{rn_r}]' \in \mathbb{R}^{n_r}$ with $n_r < n$, $f_r(x_r) \in \mathbb{R}^{n_r}[x_r]$, $B_r(x_r) \in \mathbb{R}^{n_r \times n_u}[x_r]$, $h_r(x_r) \in \mathbb{R}^{n_y}[x_r]$ and $D_r(x_r) \in \mathbb{R}^{n_y \times n_u}[x_r]$. The error system is given by

$$\dot{\chi} = \mathcal{F}(\chi) + \mathcal{B}(\chi)u, \qquad (3)$$
$$e = \mathcal{H}(\chi) + \mathcal{D}(\chi)u,$$

where

$$\mathcal{F}(\chi) = \begin{bmatrix} B(x) \\ B_r(x_r) \end{bmatrix}, \; \mathcal{B}(\chi) = \begin{bmatrix} B(x) \\ B_r(x_r) \end{bmatrix}, \chi = \begin{bmatrix} x \\ x_r \end{bmatrix},$$
$$\mathcal{H}(\chi) = h(x) - h_r(x_r), \; \mathcal{D}(\chi) = D(x) - D_r(x_r).$$

This paper aims at obtaining a reduced order model (2) such that $e \in \mathcal{L}_2[0,T]$ whenever $u \in \mathcal{L}_2[0,T]$ for $T \in [0,\infty)$. The quality of the approximant (2), in this case, is quantified by means of $\mathcal{L}_2$ gain of the error system (3). The $\mathcal{L}_2$ gain is defined as follows.

Definition 1 *[10] The error system (3) with $\chi(0) = 0$ is finite gain $\mathcal{L}_2$ stable with gain at most $\epsilon \geq 0$ if*

$$\int_0^T \|e(t)\|^2 dt \leq \epsilon^2 \int_0^T \|u(t)\|^2 dt$$

for all $u \in \mathcal{L}_2[0,T]$ and $T \in [0,\infty)$.

Throughout the paper we assume the following.

Assumption 2 $\epsilon^2 I_{n_u} - \mathcal{D}(\chi)'\mathcal{D}(\chi) \succ 0$ *for all χ.*

The following condition is sufficient for the error system to be finite gain $\mathcal{L}_2$ stable. Throughout the paper we will employ this condition when we refer to $\mathcal{L}_2$ gain stability.

Proposition 1 *[5] System (3) is finite gain $\mathcal{L}_2$ stable with gain at most $\epsilon \geq 0$ if there exists a continuosly differentiable, positive semidefinite storage function $V(\chi)$ such that*

$$\frac{\partial V(\chi)}{\partial \chi}\mathcal{F}(\chi) + \mathcal{H}(\chi)'\mathcal{H}(\chi) + \left(\frac{1}{2}\frac{\partial V(\chi)}{\partial \chi}\mathcal{B}(\chi) + \mathcal{H}(\chi)'\mathcal{D}(\chi)\right)\left(\epsilon^2 I_{n_u} - \mathcal{D}(\chi)'\mathcal{D}(\chi)\right)^{-1} \times \left(\frac{1}{2}\mathcal{B}(\chi)'\frac{\partial V(\chi)}{\partial \chi}' + \mathcal{D}(\chi)'\mathcal{H}(\chi)\right) \leq 0 \qquad (4)$$

for all $\chi \in R^{n+n_r}$.

Proof: (Short proof) By completion of square, (4) implies that system (3) is finite gain $\mathcal{L}_2$ stable with gain at most $\epsilon \geq 0$. ■

C. Sum of Squares

We define a polynomial in the form

$$p(x) = \sum_i p_i^2(x)$$

as a sum of squares (SOS) polynomial when $p_i(x)$ are polynomials. It is obvious that any polynomial which can be expressed as an SOS of other polynomials is nonnegative everywhere. One way to express an SOS equivalently is by

$$p(x) = z^T(x) M z(x)$$

where M is a positive semidefinite symmetric matrix and $z(x)$ is monomial of degree less than or equal to half of the degree of $p(x)$. For the same monomial $z(x)$ it might be possible to have similar representation with different M with M being not positive semidefinite. Thus if the intersection of $\{M \in S_n | p(x) = z^T(x)Mz(x)\}$ with $\{M \in S_n | M \succeq 0\}$ is not empty then $p(x) = z^T(x)Mz(x)$ is an SOS. Within this direction, in [6] the author showed that determining whether a polynomial can be expressed as an SOS is an LMI problem. Hence the problem of testing whether a polynomial is sum of squares becomes relatively easy as it can be computed using semidefinite programming. In view of the fact that verifying nonnegativity of a polynomial is very difficult, throughout the paper, we will relax most polynomial inequalities by replacing nonnegativity with SOS condition.

D. Estimate of the Reachable Set

Consider the inequality

$$\frac{\partial L_c(x)}{\partial x} f(x) + \frac{1}{2}\frac{\partial L_c(x)}{\partial x} B(x) B(x)' \frac{\partial L_c(x)}{\partial x}' \leq 0 \quad \forall x \qquad (5)$$

or, equivalently

$$\frac{\partial L_c(x)}{\partial x} f(x) + \frac{\partial L_c(x)}{\partial x} B(x) u \leq \frac{1}{2} u'u \quad \forall (x,u)$$

where the function $L_c(x)$ is positive definite and $L_c(0) = 0$. By setting $x(0) = 0$ we obtain

$$L_c(x(t)) \leq \frac{1}{2}\int_0^t \|u(\tau)\|^2 d\tau.$$

If we denote

$$\mathcal{R}_{c,\delta} = \left\{ x \in \mathbb{R}^n \middle| \; L_{cg}(x) \leq \frac{1}{2}\delta \right\}$$

then $\mathcal{R}_{c,\delta}$ is a set which contains all reachable states from the origin when the input u to the system

$$\dot{x} = f(x) + B(x)u, \quad x(0) = 0,$$

satisfies

$$\int_0^\infty \|u(\tau)\|^2 d\tau \leq \delta.$$

In this case we can use $\mathcal{R}_{c,\delta}$ as an estimate of the reachable set from the origin. It is important to point out that there are many choices for $\mathcal{R}_{c,\delta}$ as the function $L_c(x)$ is nonunique. Since the estimates are not unique we may consider the smallest set $\mathcal{R}_{c,\delta}$ which contains the reachable set.

E. Overview of the Approach

For linear system G with a realization $\{A, B, C, D\}$ where $A + A'$ is negative definite the error system is given by realization $\{\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}\}$ where

$$\mathcal{A} = \begin{bmatrix} A & 0 \\ 0 & A_r \end{bmatrix}, \; \mathcal{B} = \begin{bmatrix} B \\ B_r \end{bmatrix},$$
$$\mathcal{C} = \begin{bmatrix} C & -C_r \end{bmatrix}, \; \mathcal{D} = D - D_r,$$

and $\{A_r, B_r, C_r, D_r\}$ is a realization of reduced model G_r of order $n_r < n$. In this case the $\mathcal{L}_2$ gain approximation problem becomes $\mathcal{H}_\infty$ model reduction problem. Necessary and sufficient condition for the error system to have $\mathcal{H}_\infty$-norm at most $\epsilon \geq 0$ is given by the existence of a positive definite matrix $M \in \mathbb{R}^{(n+n_r)\times(n+n_r)}$ such that [3]

$$\begin{bmatrix} \mathcal{A}'M + M\mathcal{A} & M\mathcal{B} & \mathcal{C}' \\ \mathcal{B}'M & -\epsilon^2 I & \mathcal{D}' \\ \mathcal{C} & \mathcal{D} & -I \end{bmatrix} \prec 0. \tag{6}$$

Indeed the $\mathcal{H}_\infty$ model reduction problem is to find $\{A_r, B_r, C_r, D_r\}$ and positive definite M such that (6) is satisfied for a minimum value of ϵ. But this problem is not easy to solve in terms of computation as inequality (6) is not convex in the unknown variables M, A_r, B_r because of the coupling terms $M\mathcal{A}$ and $M\mathcal{B}$.

Bearing the nonconvexity of our condition in mind we introduce an approach to avoid the problem of the coupling of unknown variables M, A_r, B_r in (6). Our approach is divided into two steps:

1) Compute A_r and B_r based on an estimate of the reachability set.
2) For the given A_r and B_r, compute M, C_r and D_r which satisfy (6).

It is important to note that our approach will introduce conservatism as computation of unknowns A_r and B_r is based on an estimate instead of the exact reachability set. Moreover A_r, B_r, C_r, D_r and M are not simultaneously computed while minimizing ϵ in (6). So in this case the minimum value of ϵ obtained through this scheme is not guaranteed to be optimum.

The rest of the paper is devoted to discussing this approach. First, we will elaborate this approach for linear system. By using the same way of reasoning we will extend the use of this approach to polynomial system (1). Indeed, by Schur complement [2] and Assumption 2, the inequality

$$\begin{bmatrix} \frac{\partial V(\chi)}{\partial \chi}\mathcal{F}(\chi) & \frac{1}{2}\frac{\partial V(\chi)}{\partial \chi}\mathcal{B}(\chi) & \mathcal{H}(\chi)' \\ \frac{1}{2}\mathcal{B}(\chi)'\frac{\partial V(\chi)}{\partial \chi}' & -\epsilon^2 I_{n_u} & \mathcal{D}(\chi)' \\ \mathcal{H}(\chi) & \mathcal{D}(\chi) & -I_{n_y} \end{bmatrix} \preceq 0, \tag{7}$$

is equivalent to (4). A relaxation of (7) in terms of sum of squares is given by [9]

$$-w' \begin{bmatrix} \frac{\partial V(\chi)}{\partial \chi}\mathcal{F}(\chi) & \frac{1}{2}\frac{\partial V(\chi)}{\partial \chi}\mathcal{B}(x) & \mathcal{H}(\chi)' \\ \frac{1}{2}\mathcal{B}(\chi)'\frac{\partial V(\chi)}{\partial \chi}' & -\epsilon^2 I_{n_u} & \mathcal{D}(\chi)' \\ \mathcal{H}(\chi) & \mathcal{D}(\chi) & -I_{n_y} \end{bmatrix} w \tag{8}$$

is SOS for $w \in \mathbb{R}^{1+n_u+n_y}$ and $\chi \in \mathbb{R}^{n+n_r}$.

Yet this relaxation is not possible to verify by means of tractable computation because of the coupling of the unkowns $V(\chi)$, $f_r(x_r)$ and $B_r(x_r)$ in the terms $\frac{\partial V(\chi)}{\partial \chi}\mathcal{F}(\chi)$ and $\frac{\partial V(\chi)}{\partial \chi}\mathcal{B}(x)$. Hence we face the same type of problem like in linear system where the coupling of the unknowns renders the computation intractable. Like in linear part our approach to avoid the coupling terms for nonlinear system is to compute $f_r(x_r)$ and $B_r(x_r)$ independently from the computational scheme of $V(\chi)$, $h_r(x_r)$ and $D_r(x_r)$. To be more precise the approach for the class of nonlinear system we are considering is given as follow.

1) Compute $f_r(x_r)$ and $B_r(x_r)$ based on an estimate of the reachability set.
2) For the given $f_r(x_r)$ and $B_r(x_r)_r$, compute $V(\chi), h_r(x_r)$, $D_r(x_r)$ and minimizing ϵ which satisfy (8).

III. Reachability Based Approach

A. Linear System

We consider again

$$\dot{x} = Ax + Bu. \tag{9}$$

For an estimate of the reachable set from the origin we can select the quadratic $L_c(x) = \frac{1}{2}x'\hat{Y}^{-1}x$ where $\hat{Y}$ is a symmetric positive definite matrix of size n by n. We can write (5) in the form

$$A'\hat{Y}^{-1} + \hat{Y}^{-1}A + \hat{Y}^{-1}BB'\hat{Y}^{-1} \preceq 0$$

or equivalently

$$\hat{Y}A' + A\hat{Y} + BB' \preceq 0.$$

The estimate of the reachable set from the origin when $\int_0^\infty u(\tau)^2 d\tau \leq \delta$ is given by

$$\mathcal{R}_{c,\delta} = \left\{ x \in \mathbb{R}^n \middle| \; x'\hat{Y}^{-1}x \leq \delta \right\}.$$

Without loosing generality we can set $\delta = 1$ and we denote

$$\mathcal{R}_c = \left\{ x \in \mathbb{R}^n \middle| \; x'\hat{Y}^{-1}x \leq 1 \right\}.$$

Since $\hat{Y}^{-1}$ is a symmetric positive definite matrix the set $\mathcal{R}_c$ is a hyperellipsoid [1] where the directions and the lengths of its principal axes are defined by the eigenvectors and the inverse of square root of the eigenvalues, respectively, of matrix $\hat{Y}^{-1}$. If we denote T as an orthogonal matrix ($T'T = I$) whose columns are the normalised eigenvectors of matrix $\hat{Y}^{-1}$ then we can define a new coordinate system $z = T^{-1}x$ where its main axis coincide with the principal axis of the ellipsoid. By denoting $S = diag(\lambda_1, \ldots, \lambda_n)$ where λ_i is the eigenvalue of matrix $\hat{Y}^{-1}$ and the $i-$th column of matrix T is the eigenvector with respect to λ_i we have $\hat{Y}^{-1}T = TS$. Indeed the length of the axis with respect to the $i-$th eigenvector is equal to $1/\sqrt{\lambda_i}$.

With respect to the new coordinate system we can rewrite linear system (9) in the form

$$\dot{z} = \hat{A}z + \hat{B}u$$

where $\hat{A} = T^{-1}AT$ and $\hat{B} = T^{-1}B$. The estimate of the reachable set in the new coordinate system is given by

$$\mathcal{R}_c = \left\{ z \in \mathbb{R}^n |\ z'T'\hat{Y}^{-1}Tz \leq 1 \right\} = \left\{ z \in \mathbb{R}^n |\ z'Sz \leq 1 \right\}.$$

Suppose we order the eigenvalues in such a way that $\lambda_i \leq \lambda_j$ whenever $i \leq j \leq n$. Then from the set $\mathcal{R}_c$ we may claim that the trajectories of the system are more accumulated around the z_i-axis rather than z_j-axis for $i \leq j$. This forms the foundation of our approach where we remove the axes which are weakly reachable.

Next we partition the part of size n into two parts of size n_r and $n - n_r$ based on the following

$$z = \begin{bmatrix} z'_{[1]} & z'_{[2]} \end{bmatrix}',$$
$$\hat{A} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix},\ \hat{B} = \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{bmatrix},$$
$$S = \begin{bmatrix} \lambda_{[1]} & 0 \\ 0 & \lambda_{[2]} \end{bmatrix},$$

and the system can be expressed as

$$\dot{z}_{[1]} = \hat{A}_{11}z_{[1]} + \hat{A}_{12}z_{[2]} + \hat{B}_1u,$$
$$\dot{z}_{[2]} = \hat{A}_{21}z_{[1]} + \hat{A}_{22}z_{[2]} + \hat{B}_2u.$$

Removing the least reachable part $z_{[2]}$ we have the dynamic of our new reduced model $x_r = z_{[1]}$ given by

$$\dot{x}_r = \hat{A}_{11}x_r + \hat{B}_1u, \tag{10}$$

where

$$\hat{A}_{11} = T_1'AT_1,\ \hat{B}_1 = T_1'B.$$

with T being partitioned by

$$T = \begin{bmatrix} T_1 & T_2 \end{bmatrix}. \tag{11}$$

To sum up we have approximated the linear system (9) with another system of lower dimension given by (10) with the argument that the least reachable parts in (9) are removed from (10) while the most influential parts in (9) are preserved in (10).

B. Extension to Nonlinear System

To extend the ideas in linear system to nonlinear system we consider again (1) with all the assumptions. We assume the existence of a positive definite polynomial function $L_c(x)$ which satisfies (5). Associated with $L_c(x)$ we denote the set

$$\mathcal{R}_c = \left\{ x \in \mathbb{R}^n |\ 2L_c(x) \leq 1 \right\}. \tag{12}$$

As the function $L_c(x)$ can be nonquadratic for a nonlinear system and if we express such a function in a way like in the linear case, that is

$$L_c(x) = \frac{1}{2}x'\hat{Y}(x)^{-1}x,$$

we will have difficulties in computing the eigenvalues and eigenvector of polynomial matrix $\hat{Y}(x)^{-1}$ as it is not a constant matrix anymore. Instead of dealing with nonquadratic $L_c(x)$ to determine 'the most important' axis we introduce another quadratic function $\hat{L}_c(x) = \frac{1}{2}x'\Psi^{-1}x$ where

$$\mathcal{R}_c \subseteq \left\{ x \in \mathbb{R}^n |\ 2\hat{L}_c(x) \leq 1 \right\} = \hat{\mathcal{R}}_c.$$

Hence the set $\hat{\mathcal{R}}_c$ is also an estimate of the reachable set as $\hat{\mathcal{R}}_c$ contains $\mathcal{R}_c$. Though $\hat{\mathcal{R}}_c$ is more conservative than $\mathcal{R}_c$, it has a nice shape in a way that the set

$$\hat{\mathcal{R}}_c = \left\{ x \in \mathbb{R}^n | 2\hat{L}_c(x) = \ x'\Psi^{-1}x \leq 1 \right\}.$$

is a hyperellipsoid where the directions and the lengths of its principal axes are defined by the eigenvectors and the inverse of square root of the eigenvalues, respectively, of matrix Ψ^{-1}. The rest will follow in the same way with those in linear system where we denote T as an orthogonal matrix ($T'T = I$) whose columns are the normalised eigenvectors of matrix Ψ^{-1} and we define a new coordinate system $z = T^{-1}x$ where its main axis coincide with the principal axis of the ellipsoid. Indeed we have $\Psi^{-1}T = TS$ where $S = diag(\lambda_1, \ldots, \lambda_n)$. In line with that in linear system the transformed nonlinear system is given by

$$\dot{z} = T'f(Tz) + T'B(Tz)u, \tag{13}$$

with

$$z'T'f(Tz) = x'f(x) \leq -x'Qx = -zT'QTz. \tag{14}$$

From the fact that $T^{-1} = T'$ we have

$$\hat{Q} = T'QT \succ 0.$$

The set $\hat{\mathcal{R}}_c$ can be written in terms of new coordinate

$$\hat{\mathcal{R}}_c = \left\{ z \in \mathbb{R}^n |\ z'T'\Psi^{-1}Tz \leq 1 \right\} = \left\{ z \in \mathbb{R}^n |\ z'Sz \leq 1 \right\}.$$

By ordering $\lambda_i \leq \lambda_j$ whenever $i \leq j \leq n$ then the trajectories of the system are more accummulated around the z_i-axis rather than z_j-axis. By removing the weakly reachable parts of (13) we can truncate (13) to obtain a reduced order model of dimension $n_r < n$ in the form

$$\dot{x}_r = f_r(x_r) + B_r(x_r)u \tag{15}$$

where

$$f_r(x_r) = T_1' f(T_1 x_r), \; B_r(x_r) = T_1' B(T_1 x_r)$$

and T comes in the form (11). Removing the least reachable part from (14) and partitioning

$$\hat{Q} = \begin{bmatrix} \hat{Q}_1 & \hat{Q}_2 \\ \hat{Q}_2 & \hat{Q}_3 \end{bmatrix}$$

it follows that

$$x_r' f_r(x_r) \leq -x_r' \hat{Q}_1 x_r$$

where $\hat{Q}_1 \succ 0$ and thus the origin of the unforced truncated system (15) is globally asymptotically stable.

To reduce conservatism of the set $\hat{\mathcal{R}}_c$ we require that the set $\hat{\mathcal{R}}_c$ should be contained in as small ball B_r as possible. Therefore we need to minimize $r > 0$ such that $\mathcal{R}_c \subseteq \hat{\mathcal{R}}_c \subseteq B_r$. A sufficient condition for the required containment is given as follows.

Lemma 1 *If*

$$\left(\frac{1}{r}\|x\|_2\right)^2 \leq 2\hat{L}_c(x) \leq 2L_c(x) \tag{16}$$

for all x then $\mathcal{R}_c \subseteq \hat{\mathcal{R}}_c \subseteq B_r$.

Proof: *All x in $\mathcal{R}_c$ satisfies $2L_c(x) \leq 1$. From $2\hat{L}_c(x) \leq 2L_c(x)$ it follows that $2\hat{L}_c(x) \leq 1$. Hence x is in $\hat{\mathcal{R}}_c$. Furthermore from $\left(\frac{1}{r}\|x\|_2\right)^2 \leq 2\hat{L}_c(x)$ we have $\left(\frac{1}{r}\|x\|_2\right)^2 \leq 1$. Thus x is also in the ball B_r.* ∎

It has been already indicated that verifying nonnegativity is a hard problem. Instead the inequalities (5) and (16) can be relaxed by means of sum of squares (SOS). We summarize our approach as follows.

1) Maximize $\theta > 0$ such that

$$L_c(x) - \hat{L}_c(x) \text{ is SOS for all } x \in \mathbb{R}^n, \tag{17}$$
$$2\hat{L}_c(x) - \theta\|x\|_2^2 \text{ is SOS for all } x \in \mathbb{R}^n,$$
$$v' \begin{bmatrix} -\frac{\partial L_c(x)}{\partial x} f(x) & \frac{\partial L_c(x)}{\partial x} B(x) \\ B(x)' \frac{\partial L_c(x)}{\partial x}' & 2I_{n_u} \end{bmatrix} v \text{ is}$$
$$\text{SOS for all } x \in \mathbb{R}^n \text{ and } v \in \mathbb{R}^{1+n_u},$$

where $L_c(x)$ is a polynomial in x and $\hat{L}_c(x)$ is a quadratic polynomial in x.

2) Compute the transformation T from

$$\hat{\mathcal{R}}_c = \left\{x \in \mathbb{R}^n | 2\hat{L}_c(x) = \; x'\Psi^{-1}x \leq 1\right\},$$

and truncate the transformed system $z = T'x$ at $n_r < n$. In this case we obtain $f_r(x_r)$ and $B_r(x_r)$.

3) Compute $h_r(x_r)$, $D_r(x_r)$ and positive semidefinite $V(\chi)$, and minimize ϵ such that

$$-w' \begin{bmatrix} \frac{\partial V(\chi)}{\partial \chi}\mathcal{F}(\chi) & \frac{1}{2}\frac{\partial V(\chi)}{\partial \chi}\mathcal{B}(x) & \mathcal{H}(\chi)' \\ \frac{1}{2}\mathcal{B}(x)'\frac{\partial V(\chi)}{\partial \chi}' & -\epsilon^2 I_{n_u} & \mathcal{D}(x)' \\ \mathcal{H}(\chi) & \mathcal{D}(x) & -I_{n_y} \end{bmatrix} w \tag{18}$$

is SOS for all $x \in \mathbb{R}^n$ and $w \in \mathbb{R}^{1+n_u+n_y}$.

IV. Numerical Example

A. Example 1

Consider the system

$$\begin{aligned} \dot{x}_1 &= -x_2 - x_3 - x_1(x_1^2 + x_2^2 + x_3^2 + 1), \\ \dot{x}_2 &= x_1 - x_3 - x_2(x_1^2 + x_2^2 + x_3^2 + 1), \\ \dot{x}_3 &= x_1 + x_2 - x_3(x_1^2 + x_2^2 + x_3^2 + 1) + u, \\ y &= x_1. \end{aligned}$$

We want to compute a reduced model of order two. By feasibility test of (17) we obtain

$$L_c(x) = \hat{L}_c(x) = 27x_1^2 + 5x_2^2 + 3x_3^2 - 6x_1x_2 + 10x_1x_3 + 2x_2x_3$$

and $\theta = 2.5666$. The transformation T is given by

$$T = \begin{bmatrix} -0.2201 & 0.0256 & -0.9752 \\ -0.4151 & 0.9022 & 0.1174 \\ 0.8827 & 0.4306 & -0.1879 \end{bmatrix}$$

and truncation of the transformed system gives

$$\begin{aligned} \dot{x}_{r1} &= 1.2804x_{r2} - x_{r1}\left(x_{r1}^2 + x_{r2}^2 + 1\right) + 0.8827u, \\ \dot{x}_{r2} &= -1.2804x_{r1} - x_{r2}\left(x_{r1}^2 + x_{r2}^2 + 1\right) + 0.4306u. \end{aligned}$$

Feasibility test of (18) gives

$$\begin{aligned} V(\chi) = &\; 0.68152x_1^2 + 1.5916x_2^2 + 0.4388x_3^2 - 0.76919x_1x_2 \\ & - 0.73432x_1x_3 + 0.65377x_2x_3 + 0.53614x_{r1}^2 \\ & - 0.20907x_{r1}x_{r2} + 1.816x_{r2}^2 + 0.64768x_1x_{r1} \\ & + 1.011x_1x_{r2} + 0.56639x_2x_{r1} - 3.2278x_2x_{r2} \\ & - 0.69682x_3x_{r1} - 1.0061x_3x_{r2} \end{aligned}$$

and

$$y_r = -0.2498x_{r1} - 0.1297x_{r2}$$

with $\epsilon = 0.1014$. The response of the system and the reduced model to inputs $u = e^{-1.5t}sin(1.5t)$ and $u = e^{-3t} - e^{-1.5t}$ can be seen in Fig. 1 and Fig. 2, respectively. Though the responses are not too much in agreement, our scheme is still outperformed the one in [9] as the scheme in [9] fails to compute a reduced model of order two for the original system.

B. Example 2

The following system is taken from [9]

$$\begin{aligned} \dot{x}_1 &= -x_1 - x_2, \\ \dot{x}_2 &= 2x_1 - 3x_2 - x_2^3 + u, \\ y &= x_1. \end{aligned}$$

We want to compute a reduced model of order one. For $L_c(x)$ in quadratic form (thus $L_c(x) = \hat{L}_c(x)$) we obtain $\theta = 6.000$. For $L_c(x)$ with maximum degree of four we obtain $\theta = 6.4593$. Hence $L_c(x)$ with maximum degree of four gives a better estimate of the reachable set than that of quadratic $L_c(x)$. Increasing the maximum degree of $L_c(x)$ higher than four will give the same value of θ as in $L_c(x)$ of

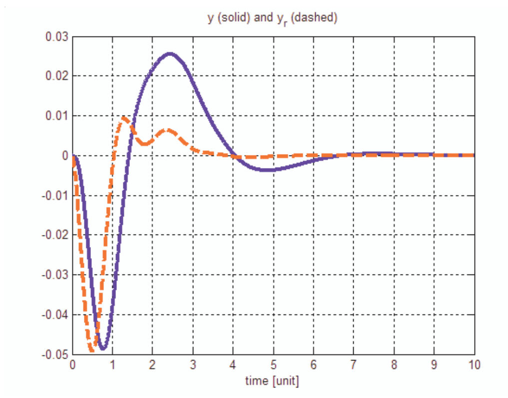

Fig. 1. Response of the output to the input $u = e^{-1.5t}sin(1.5t)$

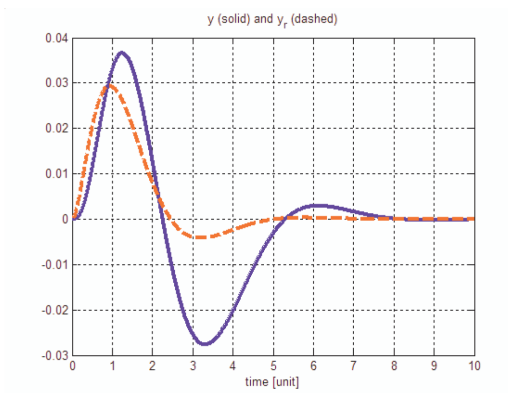

Fig. 2. Response of the output to the input $u = e^{-3t} - e^{-1.5t}$

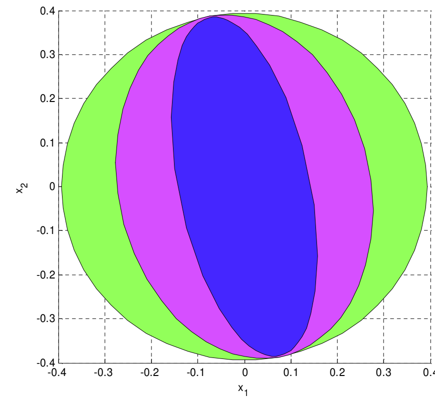

Fig. 3. Inclusion $\mathcal{R}_c \subseteq \hat{\mathcal{R}}_c \subseteq B_r$

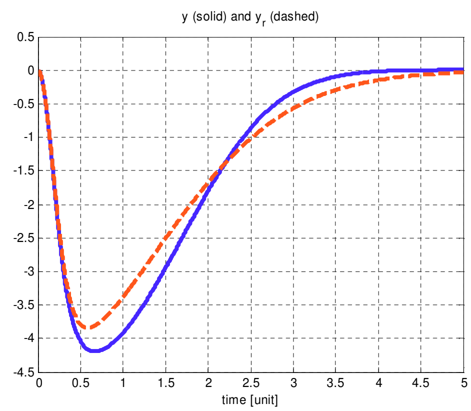

Fig. 4. Response of the output to the input $u = 50e^{-3t} - 50e^{-1.5t}$

maximum degree four. In this case we will use $L_c(x)$ with maximum degree of four where we obtain

$$L_c(x) = 24x_1^2 + 8x_1x_2 + 4x_2^2 + 6.3537x_1^4 + 0.29051x_2^4,$$
$$\hat{L}_c(x) = 6.6696x_1^2 + 1.3249x_1x_2 + 3.3572x_2^2.$$

The plot of the inclusion $\mathcal{R}_c \subseteq \hat{\mathcal{R}}_c \subseteq B_r$ where $r = \frac{1}{\theta}$ can be seen in Fig. 3. The reduced model is given by

$$\dot{x}_r = -3.1142x_r - 0.9298x_r^3 - 0.9820u,$$
$$y_r = -2.0512x_r,$$

with $\epsilon = 0.1298$. The response of the system and the reduced model to input $u = 50e^{-3t} - 50e^{-1.5t}$ can be seen in Fig. 4 which, qualitatively, is almost similar with that in [9].

V. Conclusion

We propose an approach to decouple the unknown variables in verifying a finite gain $\mathcal{L}_2$ stability condition for model reduction of a class of polynomial systems. First we seek a transformation based on an estimate of the reachability set of the system. The estimate is computed by means of sum of squares programming. The transformed system is then truncated and the truncated system will be the state space system of the reduced model. Through sum of squares programming the output of the reduced model is determined such that the error model satisfies the relaxation of the finite gain $\mathcal{L}_2$ stability condition.

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