
State Feedback Set Stabilization for a Class of Nonlinear Systems

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Abstract. A controller for a class of multiple input systems is proposed, with a subset of nonlinear positive systems as a special case. The controller is shown to obtain set stability of a certain subset of the state space. A simple example illustrates the theory.

1 Introduction

The class of nonlinear systems is very diverse, and encompasses many types of applications and behaviors. It seems apparent that for control purposes, one should focus on specific nonlinear system classes to exploit the structure of these, rather than make an all-encompassing theory for nonlinear systems.

One class of nonlinear systems that has strong structural constraints that can be exploited, is the class of *positive* systems. The typical class of nonlinear positive systems seems to be systems based on material balances, for example compartmental systems (Jacquez and Simon, 1993) and systems based on mass balances (Bastin, 1999). These system classes enjoy in addition to positivity structural constraints related to the flow of “mass” between the states. Similar structural constraints are assumed on the system class in Imsland and Foss (2002), Imsland (2002).

In this paper, the situation compared to Imsland and Foss (2002) is generalized. As an outset, we consider a (non-positive) system which have a set of positive *inventories* (which can be interpreted as mass, energy, entropy, ...). This allows more general processes than mass balance systems. Furthermore, the structural constraints are related to a general nonlinear function (rather than the sum) of the states, and the controller aims at a constant value of this nonlinear function. Hence, the “total mass”-approach used in Imsland and Foss (2002) becomes a special case of the approach used herein. Similarly to Imsland and Foss (2002) we allow multiple-input systems by dividing the states into what we call *phases*.

The considered system class is presented in Section 2, while we in Section 3 first recapitulates the necessary Lyapunov theorems before we give the main result (Theorem 3). After a brief discussion in Section 4, we present a simple example to illustrate the results.

The notation is fairly standard. $|z|_{\mathcal{A}}$ denotes the distance from z to the set \mathcal{A} , defined by $|z|_{\mathcal{A}} := \inf_{\tilde{z} \in \mathcal{A}} \|z - \tilde{z}\|$ where $\|\cdot\|$ is the Euclidean norm. Furthermore, $\mathbb{R}_+ = [0, \infty)$.

2 System class

Consider the input affine state-space system

$$\dot{x} = f(x) + g(x)u, \quad (1)$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}_+^m$. The inputs are positive and governs in some sense the inflow of the process system.

For this system, there exists of a positive vector function of “inventories” (Farsham, Viswanath and Ydstie, 1998) $\nu(x) = (\nu_1(x), \dots, \nu_p(x)) \subset \mathbb{R}_+^p$, which behave according to

$$\dot{\nu} = \frac{\partial \nu}{\partial x} f(x) + \frac{\partial \nu}{\partial x} g(x)u := F(x) + G(x)u. \quad (2)$$

At this point, note that the *positive* system $\dot{\nu} = F(\nu) + G(\nu)u$ is a special case.

We will assume that the inventories are divided into m different groups (one for each input), which for convenience will be denoted *phases*. Phase j will consist of r_j inventories (with $\sum_{j=1}^m r_j = p$), and have the control u_j associated with it (controlling the inflow to that phase). The inventories in phase j will be denoted ν^j , such that $\nu = (\nu^1, \dots, \nu^m)$. Note that $m \leq p$, such that ν^j of phase j can consist of several inventories ν_i . Corresponding to this structure, we write the vector function $F(x)$ and the matrix function $G(x)$ as

$$F(x) = (F^1(x), \dots, F^m(x)), \quad F^j(x) = (\phi_1^j(x), \dots, \phi_{r_j}^j(x)),$$

$$G(x) = \text{blockdiag}(G^1(x), \dots, G^m(x)), \quad G^j(x) = (\gamma_1^j(x), \dots, \gamma_{r_j}^j(x)).$$

For each phase, we will choose a positive scalar function $H_j(\nu^j)$ such that the control objective is that the inventories of phase j should converge to the “level set” Ω_j , given by $\Omega_j := \{z \in \mathbb{R}^{r_j} \mid H_j(z) = C_j\}$. The inventories can typically be interpreted as mass, or energy, and the functions $H_j(\nu^j)$ as total mass or energy (of a “subprocess”). In some sense, the functions $H_j(\nu^j)$ are also inventories, but we distinguish these since the control objective is related to them.

Define the scalar function $V_j(z) := \frac{1}{2} (H_j(z) - C_j)^2$, on which we impose

Assumption 1 *The sets Ω_j are compact, and there exist constants $\underline{c}^j < \bar{c}^j$ such that for all $z \in \mathbb{R}^{r_j}$,*

$$\underline{c}^j |z|_{\Omega_j} \leq V_j(z) \leq \bar{c}^j |z|_{\Omega_j}. \quad (3)$$

Remark 1. The above assumption holds (see Imsland (2002)) for example if we for a positive system (with states ν) choose the “total mass” $H_j(\nu^j) = \sum_{i=1}^{r_j} \nu_i^j$, which is used e.g. in Bastin and Praly (1999) and Imsland and Foss (2002). The positivity of the system ($\nu^j \in \mathbb{R}_+^{r_j}$) ensures in this case that Ω_j is compact, which is essential.

The system assumptions are made with respect to the set $D \subseteq \mathbb{R}^n$:

Assumption 2 *The function $F(x) : D \rightarrow \mathbb{R}^n$ is locally Lipschitz and for $x \in \{x \in D \mid H_j(\nu^j(x)) > C_j\}$, $\sum_{i=1}^{r_j} \frac{\partial H_j(\nu^j)}{\partial \nu_i^j} \phi_i^j(x) \leq 0$ and the set $\{x \in D \mid \sum_{i=1}^{r_j} \frac{\partial H_j(\nu^j)}{\partial \nu_i^j} \phi_i^j(x) = 0 \text{ and } H_j(\nu^j) > C_j\}$ does not contain an invariant set.*

This assumption means that $F(x)$ consists of *interconnection* terms (terms that have zero net contribution to $H_j(\nu^j)$, that is, “flow” between different inventories in a phase), and the rest of the terms are dominantly outflow (at least for large $H_j(\nu^j)$).

Assumption 3 *The block diagonal matrix function $G(x) : D \rightarrow \mathbb{R}^{n \times m}$ is locally Lipschitz and $\sum_{i=1}^{r_j} \frac{\partial H_j}{\partial \nu_i^j} \gamma_i^j(x) > 0$ for all $x \in D$.*

Note that further assumptions are needed to ensure that the inventories remain positive, but these are implicitly required to hold.

3 State feedback set stabilization

A Lyapunov condition for asymptotic stability of a compact set \mathcal{A} is:

Theorem 1. *The compact set \mathcal{A} is asymptotically stable (in the sense of Lyapunov) for $\dot{x} = f(x)$ if there on an open set \mathcal{O} containing \mathcal{A} exists a C^1 function V and positive constants κ_1 and κ_2 such that*

$$\kappa_1 |x|_{\mathcal{A}} \leq V(x) \leq \kappa_2 |x|_{\mathcal{A}} \quad (4)$$

$$\frac{\partial V}{\partial x} f(x) < 0 \text{ for } |x|_{\mathcal{A}} \neq 0 \quad (5)$$

on the set \mathcal{O} .

See e.g. Rouche, Habets and Laloy (1977) for a treatment of set stability. The above condition is not the most general, but sufficient for our purpose.

In some cases, it is hard to find a Lyapunov function where the derivative is negative definite. If it is semidefinite, a straightforward adoption of LaSalle’s invariance principle (similar to the Barabashin-Krasovskii Theorem) can be used:

Theorem 2. *If the conditions of Theorem 1 hold on the compact set \mathcal{D} (containing \mathcal{A}) except that (5) is replaced with*

$$\frac{\partial V}{\partial x} f(x) \leq 0 \quad (6)$$

on \mathcal{D} , and the largest invariant set contained in the set where $\frac{\partial V}{\partial x} f(x) = 0$ is contained in \mathcal{A} , then the set \mathcal{A} is asymptotically stable for $\dot{x} = f(x)$.

Recalling that asymptotic stability is equivalent to stability and convergence (Rouche et al., 1977), this result is obtained by noting that the semidefiniteness of the derivative of the Lyapunov function implies (set) stability, while convergence follows from LaSalle's invariance principle (LaSalle, 1960).

The controller we suggest for the given system class, is

$$u_j(x) = \max\left\{0, \left[\sum_{i=1}^{r_j} \frac{\partial H_j}{\partial \nu_i^j} \gamma_i^j(x) \right]^{-1} \left(-\sum_{i=1}^{r_j} \frac{\partial H_j}{\partial \nu_i^j} \phi_i^j(x) - \lambda_j (H_j(\nu^j) - C_j) \right)\right\} \quad (7)$$

where the λ_j s are positive constants. Defining $\Omega := \bigcup_{j=1}^m \Omega_j$, we make the following assumption:

Assumption 4 *There exists a (not necessarily compact) set D that is i) invariant for the dynamics (1) in closed loop with control (7), ii) has a nonempty intersection with Ω , and iii) for $x \in \Omega \cap D$, $u_j(x) > 0$.*

The stability properties can be summarized as follows:

Theorem 3. *Under Assumptions 1-4, the set $\Omega \cap D$ is asymptotically stable for the closed loop system controlled with (7). Moreover, convergence to $\Omega \cap D$ and boundedness of trajectories holds for initial conditions $x(0) \in D$.*

Proof. The set D is by Assumption 4 invariant, hence Assumptions 2 and 3 hold along closed loop trajectories. By Assumption 3, the control (7) is well defined on D .

Define the positive semidefinite function $V(x) := \sum_{j=1}^m V_j(\nu^j(x))$, with time derivative along system trajectories

$$\begin{aligned} \dot{V}(x) &= \sum_{j=1}^m [H_j(\nu^j(x)) - C_j] \dot{H}_j(\nu^j(x)) \\ &= \sum_{j=1}^m [H_j(\nu^j(x)) - C_j] \left(\sum_{i=1}^{r_j} \frac{\partial H_j}{\partial \nu_i^j} \phi_i^j(x) + \sum_{i=1}^{r_j} \frac{\partial H_j}{\partial \nu_i^j} \gamma_i^j(x) u_j(x) \right). \end{aligned}$$

For $H_j(\nu^j) \neq C_j$, we have one of the following cases:

1. If $u_j(x) > 0$, summand j is

$$[H_j(\nu^j) - C_j] \left(\sum_{i=1}^{r_j} \frac{\partial H_j}{\partial \nu_i^j} \phi_i^j(x) + \sum_{i=1}^{r_j} \frac{\partial H_j}{\partial \nu_i^j} \gamma_i^j(x) u_j(x) \right) = -\lambda_j [H_j(\nu^j) - C_j]^2 < 0.$$

2. If $u_j(x) = 0$, then summand j is $[H_j(\nu^j) - C_j] \sum_{i=1}^{r_j} \frac{\partial H_j}{\partial \nu_i^j} \phi_j^i(x)$. Thus, if $H_j(\nu^j) > C_j$, then by Assumption 2 $\sum_{i=1}^{r_j} \frac{\partial H_j}{\partial \nu_i^j} \phi_j^i(x) \leq 0$, and

$$[H_j(\nu^j) - C_j] \sum_{i=1}^{r_j} \frac{\partial H_j}{\partial \nu_i^j} \psi_j^i(x) \leq 0.$$

If $H_j(\nu^j) < C_j$, then since $u_j(x) = 0$ and $\sum_{i=1}^{r_j} \frac{\partial H_j}{\partial \nu_i^j} > 0$ by Assumption 3, we see that $\sum_{i=1}^{r_j} \frac{\partial H_j}{\partial \nu_i^j} \phi_j^i(x) \geq -\lambda_j [H_j(\nu^j) - C_j]$, which gives

$$[H_j(\nu^j) - C_j] \sum_{i=1}^{r_j} \frac{\partial H_j}{\partial \nu_i^j} \phi_j^i(x) \leq -\lambda_j [H_j(\nu^j) - C_j]^2 < 0.$$

If $H_j(\nu^j) = C_j$, $\nu^j \in \Omega_j$ and $\dot{V}_j \equiv 0$ due to Assumption 4 iii).

This holds for all summands (phases), and we conclude that $\dot{V}(x) \leq 0$. This implies that $V(x(t)) \leq V(x(t_0))$ along system trajectories. From the construction of V and Assumption 1, we see that $\|x\| \rightarrow \infty$ if and only if $V(x) \rightarrow \infty$, hence bounded $V(x(t))$ implies that $\|x(t)\|$ is bounded (stays in the compact set given by $\{x \mid V(x) \leq V(x(t_0))\}$).

We conclude by Theorem 2 that $\Omega \cap D$ is asymptotically stable, since by Assumption 1, V fulfills (4) with respect to Ω , and as shown above, $\dot{V}(x) \leq 0$ for $x \in D$. Moreover, from the above and Assumption 2, the only invariant set where $\dot{V}(x) = 0$ is (within) $\Omega \cap D$. \square

Remark 2. For positive systems, the set D can in many cases be taken as the positive orthant, $D = \mathbb{R}_+^n$. Another option can be ‘‘Lyapunov level sets’’ of the type

$$\{x \in \mathbb{R}^n \mid C_j - \underline{\beta}_j \leq H_j(\nu^j(x)) \leq C_j + \overline{\beta}_j, j = 1, \dots, m\}.$$

Similarly to De Leenheer and Aeyels (2002), we can state the following result on asymptotic stability of equilibria. It can be proved the same way as in De Leenheer and Aeyels (2002), or seen as a consequence of the theory of semidefinite Lyapunov functions (Chabour and Kalitine, 2002), hence we state it without proof:

Theorem 4. *Let the conditions of Theorem 3 hold. If the closed loop (1) with control (7) has a single equilibrium in the interior of $\Omega \cap D$ that is asymptotically stable with respect to initial conditions in $\Omega \cap D$ and attractive for all initial conditions in $\Omega \cap D$, the equilibrium is asymptotically stable for the closed loop with a region attraction (of at least) D .*

4 Discussion of controller

4.1 Connection with similar control schemes

The trained eye will see that the proposed controller is similar to feedback linearizing controllers (as, for instance, in Isidori (1995)). However, while feedback linearization linearizes the whole state space, (7) linearizes (in the unconstrained case) only the “phase dynamics” (the dynamics of the functions $H_j(\nu^j(x))$), while the total dynamics remain nonlinear. Feedback linearization requires in general solving a set of PDEs to find the right “full relative degree” output, while for the approach herein, the functions H_j are given by assumption. Furthermore, the present approach can preserve stability under some input constraints, due to the system properties.

The controller (7) is the same as proposed in Bastin and Praly (1999) (see also Bastin and Provost (2002)). Herein, a larger class of systems is studied, for instance the concept of phases allow to handle systems with multiple inputs. This idea is also explored in Imsland and Foss (2002), Imsland (2002) in the setting of positive systems, but herein the concept of “inventories” allows us to move beyond positive systems. Moreover, the approach in Imsland and Foss (2002) is generalized herein in allowing more general functions H_j .

The inventory concept is borrowed from Farsham et al. (1998), where interesting connections are tied between the first and second law of thermodynamics and system theoretic properties of “process systems”. The control problem therein is to steer the inventories to their setpoints, and this is done by assuming it is possible to manipulate the flows corresponding to *each* inventory. Only convergence of the inventories are considered, the underlying system behavior is not examined. One major difference to the controller developed herein, is that we do not assume a one-to-one relationship between inventories and manipulated flows.

Constraints are considered in Farsham et al. (1998) by augmenting the controller to a PID type controller with antireset windup. The “original” controller is in this setting a P-controller, in the same way as the one proposed herein (interpreting the λ_j s as gains).

4.2 Outflow controlled systems and upper constraints

In view of Assumption 3 (and the sign of H_j), it can be said that the system class treated herein is *inflow* controlled (with respect to H_j). However, systems that in a corresponding sense are *outflow* controlled (or having a mixture of inflow or outflow controlled phases), can be treated in a similar way. This is done in Imsland and Foss (2002) for positive systems with $H_j(\nu^j) = \sum_{i=1}^{r_j} \nu_i^j$. In addition to more rigid conditions on the “uncontrolled” flows, one must then in general have an upper saturation on the control to ensure that the inventories remain positive.

5 Example

For mass balance systems where $H_j(\nu^j)$ is the sum of the phase states (masses), $H_j(\nu^j) = \sum_{i=1}^{r_j} \nu_i^j$, several examples can be found in Imsland (2002), for instance the stabilization of a gas lifted oil well (Imsland and Foss, 2002).

Here we will consider an example which is a (slightly modified) system taken from De Leenheer and Aeyels (2002), which illustrate another type of H_j for a positive system (states are inventories). The system equations are

$$\dot{x}_1 = -x_1x_2 + x_2^2 - x_1 + u \quad (8)$$

$$\dot{x}_2 = -x_1x_2 + x_1^2 + u. \quad (9)$$

We choose $H(x) = \frac{1}{2}(x_1^2 + x_2^2)$, which gives $\frac{\partial H}{\partial x}F(x) = -x_2^2$ and $\frac{\partial H}{\partial x}G(x) = x_1 + x_2$. The assumptions 2-3 hold on $D = \mathbb{R}_+^2$, hence by Theorem 3 the control law given by (7) globally (on the positive orthant) stabilizes the set $\frac{1}{2}(x_1^2 + x_2^2) = C$, for any positive C .

Furthermore, it is rather easy to show that on the set $H(x) = C$ the system has a single equilibrium at $x_1^* = \frac{\sqrt{1+16C}-1}{4}$, $x_2^* = \sqrt{2C - (x_1^*)^2}$ which is asymptotically stable with respect to the set $H(x) = C$. By Theorem 4, this equilibrium is stable for all initial conditions in \mathbb{R}_+^2 .

A simulation confirming these results is shown in Figure 1. Note the initial input saturation.

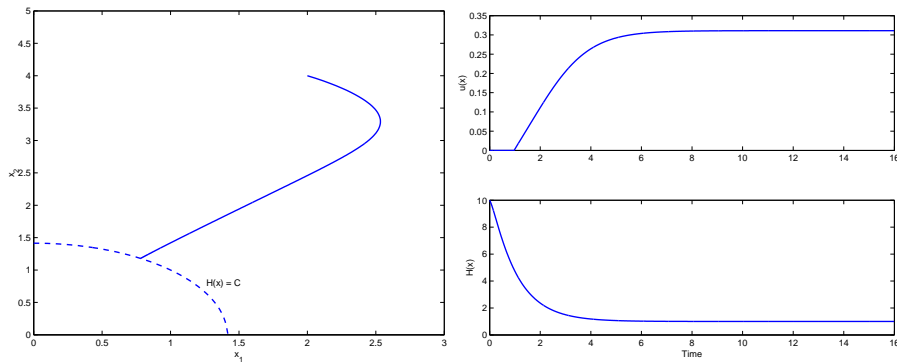


Fig. 1. To the left, a phase-plot, to the right $u(x)$ and $H(x)$. The simulation had initial condition $(2, 4)$, and the parameters $C = 1$ and $\lambda = 1$.

6 Concluding remarks

The concept of set stability is used to demonstrate the stability properties of a certain positive controller, for a special system class. One important restriction of the system class is the assumptions that ensure that the “Lyapunov

function” used in the proof of the main result is decreasing when the input saturates. According to Bastin and Praly (1999), this condition can be seen as a dissipativity property of the system.

Acknowledgment

The NTNU Natural Gas Research Center is acknowledged for financial support.

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