MULTI-MODEL BASED UNCERTAINTY AND ROBUST CONTROL DESIGN

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Abstract: This paper shows how to construct multi-models that represent partially unknown nonlinear systems and also how to perform a control design based on such a representation. New results related to a previously developed LMI/BMI-based controller design approach are presented. The controller design approach provides a piecewise affine state-feedback and allows for inclusion of control input- and state constraints. The applicability of the approach is evaluated on a simplified model of a van der Vusse reactor.

Keywords: Difference inclusion, quadratic stability, local BMI algorithm, outer approximation

1. INTRODUCTION

When designing controllers it is advantageous to be able to take explicitly into account knowledge of the model uncertainty. Model uncertainty can be formulated in different ways; the most common being frequency-based model uncertainty and parameter-based uncertainty. Frequency-based model uncertainty is used to describe sets of transfer functions. There exist powerful techniques like H_∞ design and structured singular values for robust control analysis and design based on this model description. Parameter-based uncertainty is usually used to construct sets of state-space models in which model parameters are defined to belong to a parameter set. In this paper we focus on parameter based uncertainty.

Multi-models are usually used to design nonlinear models as a convex combination of local models. The local models are typically valid in different parts of the region of interest. The collection of local model are combined to construct a model that is valid in the region of interest, see e.g. (Murray-Smith and Johansen, 1997), (Takagi and Sugeno, 1985).

An alternative use of multi-models is to describe model uncertainty. This was proposed in (Slupphaug and Foss, 1999), (Slupphaug, 1998). The motivation for exploiting this model uncertainty class is the fact that it can be used to construct affine state-feedback controllers with robust stability guarantees, cf. (Slupphaug and Foss, 1999), (Slupphaug, 1998). Moreover, the synthesis method can include state- and control input constraints.

In this paper we present new theoretical results related to application of the mentioned synthesis method. These results have been motivated by application of the synthesis method to various examples. Herein we provide an application to a simplified van der Vusse reactor model with parameter uncertainty. The outline is as follows: We first formalize the construction of the multi-model uncertainty class given a nonlinear model description with parameter uncertainty. Second, we present a local solution approach to the bilinear matrix inequality (BMI) feasibility problem associated with the synthesis method. Finally, we provide the van der Vusse example followed by a discussion on the revealed *strengths* and *weaknesses* of the methodology.

2. MULTI-MODEL BASED UNCERTAINTY

The robust control design method given in (Slupphaug and Foss, 1999) assumes that the uncertain constrained nonlinear system is, when $x_k \in X_i^C \subset \mathbb{R}^n$, given by

$$x_{k+1} \in \mathcal{M}_i(x_k,u_k,k)\ \forall u_k \in U_m, k \in \mathbb{N},$$

where $U_m \subset \mathbb{R}^m$ and $X_m := \cup_{i\in\{1,\dots,N_c\}} X_i^C$ denote the model validity sets, and X_i^C denotes a so-called cluster. The clusters have the property that they partition X_m. The $\mathcal{M}_i$s are given by that $\forall i \in \{1,\dots,N_c\}$

$$\begin{aligned}&\mathcal{M}_i(x,u,k) := \{\hat{x} | \hat{x} = A_k x + B_k u + c_k \\ &\text{for some } (A_k,B_k,c_k) \in \\ &\mathrm{Co}\{(A_{i_1},B_{i_1},c_{i_1}),\dots,(A_{i_{N_m^i}},B_{i_{N_m^i}},c_{i_{N_m^i}})\} \\ &\text{where all } i_j \in \Lambda_i \subset \{1,\dots,N_m\}\}\end{aligned}$$

and if $0 \in \text{Closure}\, X_i^C$, the associated c_is must be zero. (This last assumption means that the equilibrium control input for the state setpoint is assumed to be known.) We assume that this is the case for the first N_c^o out of the N_c clusters. N_m^i denotes the number of local models defining the dynamics (or, having support) in cluster i, while N_m denotes the total number of local models.

Next, we describe one way of finding the local affine models for a class of nonlinear systems having uncertain parameters.

2.1 *A piecewise affine envelope of a family of nonlinear systems*

Assume that

$$x_{k+1} \in \mathcal{F}(x_k, u_k, \Theta)$$

where

$$\mathcal{F}(x, u, \Theta) := \{\hat{x} | \hat{x} = f(x, u, \theta) \text{ for some } \theta \in \Theta\}$$

and f and Θ are assumed to be known. Also, assume that $\forall j \in \{1, \ldots, n\}$, $i \in \{1, \ldots, N_c\}$, $(x, u, \theta) \in X_i^C \times U_m \times \Theta$

$$\begin{aligned} A_{0j}^i x + B_{0j}^i u + c_{0j}^i &\le f_j(x, u, \theta) \\ &\le A_{1j}^i x + B_{1j}^i u + c_{1j}^i, \end{aligned} \tag{1}$$

where 0 encodes 'lower', 1 encodes 'upper', and $\forall \Psi \in \{A, B\}, l \in \{0, 1\}, i \in \{1, \ldots, N_c\}, j \in \{1, \ldots, n\}$ Ψ_{lj}^i are row vectors with appropriate dimensions, and the c_{0j}^is and c_{1j}^is are scalars. Next, let

$$\Lambda_i = \{2^n(i-1)+1, \ldots, 2^n i\}$$

and $\forall l \in \{1, \ldots, N_c 2^n\}$

$$\Psi_l = \begin{bmatrix} \Psi^p_{B(l-1,1)1} \\ \vdots \\ \Psi^p_{B(l-1,n)n} \end{bmatrix}, \ \forall \Psi \in \{A, B, c\}, \tag{2}$$

where p is given by that $l \in \Lambda_p$ and $B(l, q)$ is the qth digit of the binary representation of the positive integer l (e.g. $B(8, 3) = 0$ since $8_{10} = 100_2$). Now, it can be shown that $\forall i \in \{1, \ldots, N_c\}$, when $x \in X_i^C$

$$\mathcal{F}(x, u, \Theta) \subset \mathcal{M}_i(x, u, k), \ \forall u \in U_m, k \in \mathbb{N}.$$

This means that the uncertain system associated with $\mathcal{F}$ is contained in the uncertain system associated with the $\mathcal{M}_i$s. Thus, if we can stabilize some point x_s of the latter uncertain system, we will at the same time stabilize the same point for $\mathcal{F}$.

To summarize, we find n upper bounding and n lower bounding scalar affine functions in each of the N_c clusters, cf. (1). Then we combine these as described in (2) to obtain the 2^n different A's, B's, and c's that constitute the uncertainty model in this cluster. Note that the number of affine bounding models grows exponentially with the state dimension n, and linearly with the number of clusters.

We will now show that finding the $N_c n$ upper bounding and $N_c n$ lower bounding scalar functions can be formulated as $2N_c n$ linear programs or $2N_c n$ sequences of linear programs. We only show the upper bounding, as the lower bounding is similar.

The affine upper bounding function for state equation j in cluster i should approximate as closely as possible the nonlinear uncertain function from above. Let $\forall j \in \{1, \ldots, n\}$

$$g_j(x, u) := \max_{\theta \in \Theta} f_j(x, u, \theta) \ \forall (x, u) \in X_m \times U_m$$

then this can be transformed into the following semi-infinite optimization problem

$$\min_{A_{1j}^i, B_{1j}^i, c_{1j}^i} \left(\begin{array}{c} \displaystyle\int \cdots \int_{X_i^C \times U_m} (A_{1j}^i x + B_{1j}^i u + c_{1j}^i \\ -g_j(x, u)) dx du \end{array} \right) \tag{3}$$

subject to $\forall z \in Z := X_i^C \times U_m$

$$h(z) := A_{1j}^i x + B_{1j}^i u + c_{1j}^i - g_j(x, u) \ge 0$$

where $z = (x, u)$. Now, assuming the multiple integral in (3) exists, this cost function is linear in the parameters. However, we have an uncountable number of linear constraints (parameterized in z). By using the following well known facts from calculus we can conservatively transform this uncountable number of linear constraints into a finite number of linear constraints. Let $\tilde{h} \in C^2$ and Z convex, then for all $v_1, w_1, v_2, w_2 \in Z$

$$\begin{aligned} \tilde{h}(v_1) = \tilde{h}(w_1) &+ \nabla \tilde{h}(w_1)^T \Delta w_1 \\ &+ \frac{1}{2} \Delta w_1^T \nabla^2 \tilde{h}(w_1 + t_1 \Delta w_1) \Delta w_1 \end{aligned} \tag{4}$$

$$\nabla \tilde{h}(v_2) = \nabla \tilde{h}(w_2) + \int_0^1 \nabla^2 \tilde{h}(w_2 + \tau \Delta w_2) \Delta w_2 d\tau \tag{5}$$

for some $t_1 \in (0, 1)$. $\Delta w_i := v_i - w_i$ for $i \in \{1, 2\}$. In particular this holds if

$$v_1 = z, \ w_1 = w, \ v_2 = w, \ w_2 = p$$

where $z, w, p \in Z$. By substituting (5) into (4) we get that for all $z, w, p \in Z$

$$\begin{aligned} &\tilde{h}(w) + \nabla \tilde{h}(p)^T (z - w) - \tilde{h}(z) = \\ &(z - w)^T \int_0^1 \nabla^2 \tilde{h}(w + \tau(w - p)) d\tau (w - p) \\ &\quad - \frac{1}{2}(z - w)^T \nabla^2 \tilde{h}(w + \tilde{t}_1 (z - w))(z - w) \end{aligned}$$

where $t_1 = \tilde{t}_1$ renders (4) true. Meaning that for all $z, w, p \in Z$

$$\left|\tilde{h}(w) + \nabla\tilde{h}(p)^T(z-w) - \tilde{h}(z)\right| \leq$$
$$\gamma(\|z-w\|_\infty \|w-p\|_\infty + \frac{1}{2}\|z-w\|_\infty^2) \quad (6)$$

if $\left\|\nabla^2\tilde{h}(z)\right\|_\infty \leq \gamma$ for all $z \in Z$.

Now, define a finite set (grid) $\mathcal{G}(\varepsilon)$ that satisfies $\forall z \in Z \exists \tilde{z} \in \mathcal{G}(\varepsilon)\ \|z-\tilde{z}\|_\infty \leq \frac{1}{2}\varepsilon$ and $Co\mathcal{G}(\varepsilon) \supseteq Z$. Note that e.g. a uniform ε-grid on a hypercube satisfies this.

Lemma 1. Assume $g_j \in C^2$, Z bounded and convex, and $\left\|\nabla^2 g_j(z)\right\|_\infty \leq \gamma$ for all $z \in Z$. Also assume[1] that for all $n+1$-tuples $\{w_1,\dots,w_{n+1}\} \subseteq \mathcal{G}(\varepsilon)$ with distinct elements that $\forall k \in \{1,\dots,n+1\} \exists p \in Co\{w_1,\dots,w_{n+1}\}\ \forall i \in \{1,\dots,n+1\}$

$$\nabla g_j(p)^T(w_i - w_k) = g_j(w_i) - g_j(w_k), \quad (7)$$

Then it holds that

$$h(z) \geq \frac{5}{8}\gamma\varepsilon^2\ \forall z \in \mathcal{G}(\varepsilon) \Longrightarrow h(z) \geq 0\ \forall z \in Z.$$

Proof. Since Z is bounded a grid exists. Take an arbitrary $z \in Z$. Clearly, $z \in Co\{w_1,\dots,w_{n+1}\}$ for some $n+1$-tuple in $\mathcal{G}(\varepsilon)$ with distinct elements. Now, in (6), let $w = w_k$ in $\mathcal{G}(\varepsilon)$ be the grid point closest to z (measured in ∞-norm), let $\tilde{h} = g_j$, and let p be such that it satisfies (7). Note that $\|z-w_k\|_\infty \leq \frac{1}{2}\varepsilon$ while $\|w_k - p\|_\infty \leq \varepsilon$. Thus, $\left|g_j(w_k) + \nabla g_j(p)^T(z-w_k) - g_j(z)\right| \leq \frac{5}{8}\gamma\varepsilon^2$ or $g_j(w_k) + \nabla g_j(p)^T(z-w_k) + \frac{5}{8}\gamma\varepsilon^2 - g_j(z) \geq 0$. Due to the assumption on the l.h.s. of the implication in the lemma and (7), it holds that $[A^i_{1j} B^i_{1j}]z + c^i_{1j} \geq g_j(w_k) + \nabla g_j(p)^T(z-w_k) + \frac{5}{8}\gamma\varepsilon^2$ and the result follows. ■

This result may be utilized by choosing an $M = 5/8\gamma\varepsilon^2$ solve for ε and then solve the associated finite LP. One may also iterate on ε until some stop criterion is satisfied, resulting in a sequence of finite LPs.

3. ROBUST CONSTRAINED STABILIZATION

3.1 *Piecewise Affine State-Feedback*

We finitely parameterize the state-feedback, $u(x)$, as a piecewise affine state-feedback. With the cluster containing the origin and the clusters which closure contains the origin, we associate a linear state feedback

$$u(x) = K_l x \text{ when } x \in X_l^C,\ l \in I_{N_c^o}. \quad (8)$$

With all the other clusters we associate an affine state feedback, i.e. for $l \in \{N_c^o + 1,\dots,N_c\}$

$$u(x) = K_l x + k_l \text{ when } x \in X_l^C. \quad (9)$$

[1] We believe that this assumption is true under mild assumptions on g_j.

Remember that the clusters form a partition of X_m, so the above defined piecewise affine state-feedback is indeed well defined.

It should be noted that there is, in principle, no problem associating the piecewise state-feedback with a different partitioning of X_m than the one associated with the clusters. For reasons of clarity, however, we restrict the piecewise affine state-feedback to be associated with the clusters.

3.2 *BMI for Robust Stabilization*

In this subsection we present the essential results from (Slupphaug and Foss, 1999) in a very condensed form (due to page constraints).

Theorem 1 (Slupphaug and Foss, 1999) states sufficient quadratic stability conditions for the origin in terms of feasibility of an LMI/BMI-problem. The conditions thus guarantee the decrease, in the different clusters, of a Lyapunov function $x \mapsto x^T P x$ ($P = P^T > 0$) along all possible closed-loop trajectories which can be generated by plants in $\mathcal{M}$ under the state-feedback (8)-(9). Further, the conditions guarantee a region of attraction $\{x | x^T P x = \alpha,\ \alpha > 0\}$ that is within X_m and outside a prespecified smallest acceptable region of attraction $\{x | x^T R_A x \leq 1, R_A > 0\}$. Theorem 2 (Slupphaug and Foss, 1999) extends Theorem 1 to include control input constraints as LMIs (and also, less conservatively, as very low complexity BMIs).

3.3 *Solving the BMI problem*

In (Slupphaug and Foss, 1999) we gave a global branch-and-bound based algorithm for solving the BMI feasibility problems in Theorems 1 and 2 mentioned above. Herein, we present a local approach.

The feasibility problem(s) can be efficiently locally approached by using the algorithm developed in (El Ghaoui *et al.*, 1997). To get in a position to apply that algorithm, we recall from (Slupphaug and Foss, 1999) that the BMI origins from the nonlinear matrix inequality $S \leq P^{-1}$. Now, certainly $S \leq P^{-1}$, if $S = (\eta P)^{-1}$ and $\eta > 1$. By replacing the matrix inequality $S \geq (\eta P)^{-1}$ with the LMI

$$\begin{bmatrix} S & I \\ I & \eta P \end{bmatrix} \geq 0 \quad (10)$$

where $\eta > 1$ is a given constant, we may use a variant of Algorithm 1 in (El Ghaoui *et al.*, 1997). Algorithm 1 is devised to aim at saturating the constraint (10) giving $S = (\eta P)^{-1}$ and hence a solution to our controller design problem. However, in the present context where only an inequality need to be satisfied (rather than the equality

$S = P^{-1}$ in (El Ghaoui *et al.*, 1997)) we may find a solution prior to convergence of the iterative algorithm. Checking $S_k \leq P_k^{-1}$ at Steps 1 and 3 is what differ Algorithm 1' below from Algorithm 1.

Algorithm 1':

Step 1 Find a feasible point P_0, S_0 for the LMIs in Theorem 1 or 2 (with control constraints represented by LMIs) and (10). If there are none the problem is guaranteed infeasible[2], exit. If $S_0 \leq P_0^{-1}$, done. Otherwise, set $k = 0$.

Step 2 Solve the LMI problem $\mathcal{P}_k$

$$t_k := \min \operatorname{Tr}(P_k S + S_k P)$$

subject to all the LMIs in Theorem 1 or 2 in addition to the LMI (10) to find P_{k+1} and S_{k+1}.

Step 3 If $S_{k+1} \leq P_{k+1}^{-1}$, done. Otherwise, set $k = k + 1$ and go to Step 2.

Theorem 2. The sequence $(t_k)_{k \geq 0}$, is bounded below by $2\frac{n}{\eta}$ and nonincreasing. Thus, $(t_k)_{k \geq 0}$ converges to some value $t_{\text{opt}} \geq 2\frac{n}{\eta}$. Equality holds if and only if $SP = \frac{1}{\eta} I$.

Proof. See (El Ghaoui *et al.*, 1997). ■

Although we find feasible solutions to the controller design problem for a wider range of input data (e.g. for more $R_A > 0$) when using the global approach mentioned above, computations show that Algorithm 1' typically finds a feasible solution to the controller design problem for a comparable range of input data. This is in accordance with the computations reported in (El Ghaoui *et al.*, 1997). In addition, a solution is, if one is found, typically found must faster than is the case when using the global approach. Thus, this local algorithm seems to provide a good alternative for problems with a 'high' number of states (more than, say, 3).

4. APPLICATION TO A SIMPLIFIED VAN DER VUSSE MODEL

Next we apply the robust controller synthesis method described in (Slupphaug and Foss, 1999), (Slupphaug, 1998) combined with the local algorithm described above and an early, ad-hoc, version of the result in Section 2.1 to a van der Vusse reactor example. The CSTR has the following reaction scheme:

$$A \overset{k_1}{\rightarrowtail} B \overset{k_2}{\rightarrowtail} C$$
$$2A \overset{k_3}{\rightarrowtail} D$$

Substance A (in the feed) reacts to substance B (product) which again reacts to substance C (unwanted). Also substance A reacts to substance D (unwanted). A typical model of the reactor dynamics includes 4 states: The concentrations of substances A and B, the reactor temperature, and the temperature in the reactor cooling jacket. The original problem, described in (Chen *et al.*, 1995), includes both concentration and temperature control of the reactor.

Next, we develop a system model.

4.1 *System model*

We simplify the original problem defined in (Chen *et al.*, 1995) by assuming perfect temperature control. Using forward Euler discretization, the discrete-time nonlinear model of the system becomes:

$$\begin{aligned} c_{k+1}^A &= c_k^A + h((u_k + u_S)(c_{A0} - (c_k^A + c_S^A)) \\ &\quad - k_1(c_k^A + c_S^A) - k_3(c_k^A + c_S^A)^2) \\ c_{k+1}^B &= c_k^B + h(-(u_k + u_S)(c_k^B + c_S^B) \\ &\quad + k_1(c_k^A + c_S^A) - k_2(c_k^B + c_S^A)) \end{aligned}$$

where u_k denotes the inverse of the detention time and h denotes the sampling period. The rest of the notation is obvious knowing that the states and the input u_k are deviation variables from an equilibrium denoted (c_S^A, c_S^B, u_S) defined by the nominal values for the reaction rates (cf. Table 1) and $h = 0.005$ hours. The offset values are: $c_S^A = 2.14\frac{1}{\text{mol}}$, $c_S^B = 1.09\frac{1}{\text{mol}}$ and $u_S = 14.19\frac{1}{\text{hour}}$.

The system is not linear in the control variable. Since the controller design method to be used handles nonlinearities in the input conservatively, it is advocated in (Slupphaug and Foss, 1999) to introduce the one-step delayed control input as a state to alleviate this conservatism. The smaller the sample period the more effective this remedy will be for removing the conservatism. Denoting the delayed control input u_k^d, we get

$$\begin{aligned} c_{k+1}^A &= c_k^A + h((u_k^d + u_S)(c_{A0} - (c_k^A + c_S^A)) \\ &\quad - k_1(c_k^A + c_S^A) - k_3(c_k^A + c_S^A)^2) \\ c_{k+1}^B &= c_k^B + h(-(u_k^d + u_S)(c_k^B + c_S^B) \\ &\quad + k_1(c_k^A + c_S^A) - k_2(c_k^B + c_S^A)) \\ u_{k+1}^d &= u_k \end{aligned} \tag{11}$$

Next, the model validity sets X_m and U_m are chosen as follows

$$\begin{aligned} X_m &:= [-1.0, 1.0] \times [-0.2, 0.2] \times [-10, 10] \\ U_m &:= [-10, 10] \end{aligned}$$

(11) is the system we will build the multimodel-based uncertainty class for.

[2] This follows since it can be shown that if there is a solution to the LMIs in Theorem 1 or 2 with $S \leq P^{-1}$ then $S = P^{-1} \geq (\eta P)^{-1}$ also solves the problem.

	$k_1 [\frac{1}{\text{hour}}]$	$k_2 [\frac{1}{\text{hour}}]$	$k_3 [\frac{1}{\text{Mol hour}}]$
Upper	15.2028	15.2028	2.3533
Nominal	14.7445	14.7445	2.2850
Lower	14.2863	14.2863	2.2168

Table 1. Reaction rates.

Note that only the nominal system, i.e. $(k_1, k_2, k_3) = (14.7445, 14.7445, 2.2850)^T$, has its equilibrium point at the origin. Due to parameter variations we do not know the steady-state value for the control variable. This is a violation of one of the assumptions in the theory for the control design. It will be commented on later.

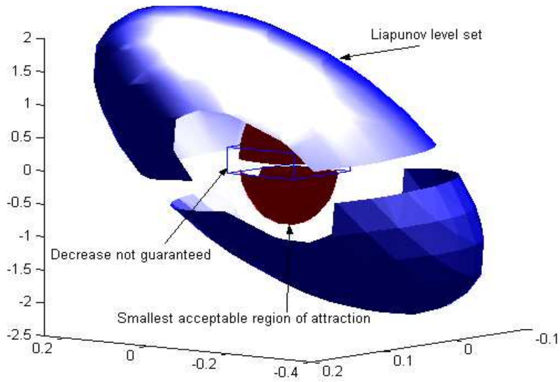

Fig. 1.

4.2 *Multimodel-based uncertainty class development*

Next, we will generate a piecewise affine envelope of the uncertain system given by (11) and Table 1. The clusters are chosen to be boxes and defined by the following partitioning of the axes:

$$\begin{aligned} c^A &\in [\,-1\ \ -0.08\,] \cup [\,-0.08\ \ 0\,] \cup [\,0\ \ 0.08\,] \\ &\quad \cup [\,0.08\ \ 1\,] \\ c^B &\in [\,-0.2\ \ -0.04\,] \cup [\,-0.04\ \ 0\,] \cup [\,0\ \ 0.04\,] \\ &\quad \cup [\,0.04\ \ 0.2\,] \\ u^d &\in [\,-10\ \ -0.2\,] \cup [\,-0.2\ \ 0\,] \cup [\,0\ \ 0.2\,] \\ &\quad \cup [\,0.2\ \ 10\,] \end{aligned}$$

Hence, the allowable set for each state is divided into 4 subsets This gives rise to $N_c = 4 \times 4 \times 4 = 64$ clusters. The closure of 8 clusters includes the origin ($N_c^o = 8$). In these clusters we develop linear models by outer approximating the *nominal* nonlinear model (we need the c's to be zero here). Although the equilibrium input is dependent on the parameters and hence the outer approximation is not valid in the clusters bordering the origin, we note that the theory guarantees that the smallest Lyapunov level set containing the clusters bordering the origin is going to be 'close to' an invariant in the sense that in the limit, as the sampling period approaches zero, it gets an invariant for the uncertain system.

As shown in (2) it is generally necessary to develop $2^n = 8$ affine models within each cluster to make sure that (11) is included. However, since the last state equation is linear and without uncertainty, we can in this case do with $2^2 = 4$ affine models within each cluster. Furthermore, since the control input enters linearly without uncertainty, we may take $Z = X_i^C$ and remove the B_{0j}^is and B_{0j}^is in the associated linear programming problems. (The B_{0j}^is and B_{0j}^i are known and hence they need not be searched for, one just insert them afterwards as shown in (2).) By this the total number of affine models for the whole state space is $64 \times 4 = 256$ models.

4.3 *Computing the feedback controller*

The aim is now to compute a piecewise affine state feedback controller with stability guarantees. Within the part of the state-space that consists of the 8 clusters that contain the origin in their closure, the controller is piecewise linear. This implies that the design problem involves computing $(64-8) \times 4 + 8 \times 3 = 248$ control parameters, and 6 parameters to define the Lyapunov function plus a lot of $\mathcal{S}$-procedure variables (Slupphaug, 1998) adding up to a total of 375 variables.

Each cluster is outer approximated by an ellipsoid (apart from the ones bordering the origin which are approximated by their respective orthants), and the state constraints and control input constraints as intersections of ellipsoids.

The resulting design problem involves 455 linear matrix inequalities and one bilinear matrix inequality. This non-convex feasibility problem is solved by the use of the local approach as described in Section 3.3. The design problem is, when a solution is found, typically solved in a couple of minutes on a 450 MHz Pentium II (256 Mb RAM).

4.4 *Results*

The obtained estimate of the region of attraction as well as the used approximations to the clusters is shown in Figures 1 and 2, respectively. Note that the clusters are not completely covered by the ellipsoids meaning that the required decrease in the Lyapunov function may not be satisfied in the whole cluster. When we tried to cover more of the clusters we did not find a feasible solution to the associated BMI problem. However, simulations show that the found Lyapunov function is decreasing along trajectories also in the non-covered parts of the clusters.

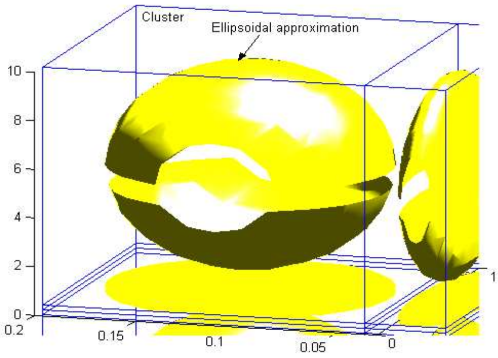

Fig. 2.

5. DISCUSSION AND CONCLUSIONS

In this work we have applied earlier developed theory together with some new results to a simplified van der Vusse reactor example. This has revealed strengths and weaknesses of the proposed methodology.

The main strength of the methodology is the ability to compute a robustly stabilizing piecewise affine control law for a nonlinear uncertain system with control input and state constraints. Moreover, we have shown that this is possible for a system with a quite complex uncertainty description even if the original problem was simplified by assuming ideal temperature control.

The theory handles nonlinearities in the control input conservatively. This was solved by adding a delayed control input as an extra state. This is not an entirely satisfactory solution since the control input is unnecessarily delayed.

The outer approximation of the cluster gives rise to conservativeness in the sense that we require the decreasing Lyapunov condition to be satisfied within a larger set than required by the theory. In the example this resulted in an infeasible design problem. To obtain a feasible solution it was necessary to reduce the outer approximating ellipsoids, cf Figure 2. Hence, the ellipsoids did not completely cover the required set. To circumvent this problem tighter outer approximations by e.g. unions of ellipsoids is one possible extension that has not been implemented yet but is described in (Slupphaug, 1998).

The background theory requires a known equilibrium point. This was not the case for the example in this study. This implies that the theory does not hold for the 8 clusters that contain the origin in their closure, cf. Figure 1. Hence, the inclusion of integral action of some sort is an obvious issue for future work.

The quite complex uncertainty description produced the need for an efficient algorithm to compute the bounding affine models since it was necessary to compute bounding models in all 64 clusters This was done by formulating this model design problem into a linear programming problem. The theoretical foundation for this is given in Lemma 1.

Computing the feedback controller involves the solution of a large LMI/BMI problem. A local solution approach, cf. Theorem 2, was suggested and the efficiency of the method was demonstrated on the example.

As for other future work, it is of interest to try to avoid the exponential growth in the number of local models, this may result in a more conservative embracing multi-model based uncertainty class. To counteract this increase in conservatism as well as the conservatism associated with allowing unbounded time-variation in the model, one may look into a less conservative class of Lyapunov functions, e.g. parameter varying Lyapunov functions.

6. ACKNOWLEDGMENTS

The authors acknowledge Mr. Oddvar Grønning for the software implementation of the uncertainty model generation, and the useful comments from Mr. Lars Imsland during the preparation of this paper.

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